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(*b*. Prague, Czechoslovakia, 5 October 1781; *d*. Prague, 18 December 1848)

Philosophy, mathematics, logic, religion, ethics.

Bolzano was born in one of the oldest quarters of Prague and was baptized Bernardus Placidus Johann Nepomuk. His mother, Caecilia Maurer, daughter of a hardwre tradesman in Prague, was a pious woman with an inclination to the religious life. At the age of twenty-two she married the elder Bernard Bolzano, an Italian immigrant who earned a modest living as an art dealer. The father was a widely read man with an active social conscience, and felt responsible for the well-being of his fellow men. He put his ideas into practice and took an active part in founding an orphanage in Prague.

Bernard was the fourth of twelve children, ten of whom died before reaching adulthood. Of delicate health, he had a quiet disposition, although he was easily irritated and very sensitive to injustice.

From 1791 to 1796 he was a pupil in the Piarist Gymnasium, and in 1796 he entered the philosophical faculty of the University of Prague, where he followed courses in philosophy, physics, and mathematics. His interest in mathematics was stimulated by reading A. G. Kästner's *Anfangsgründe der Mathematik*, because Kästner took care to prove statements which were commonly understood as evident in order to make clear the assumptions on which they depended.

The benevolence of the environment in which Bernard Bolzano was reared, both at home and in school, influenced his entire life. In fact, he raised to the supreme principle of moral conduct the precept always to choose that action, of all possible actions, which best furthers the commonweal.

After having finished his studies in philosophy in 1800, Bolzano entered the theological faculty. These studies did not strengthen his belief or resolve his doubts concerning the truth and divinity of Christian religion, but he found a solution in his professor's statement that a doctrine may be considered justified if one is able to show that faith in it yields moral profit. This made it possible for Bolzano to reconcile religion with his ethical views and to consider Catholicism the perfect religion.

In 1805 Emperor Franz I of Austria, of which Czechoslovakia was then a part, decided that a chair in the philosophy of religion would be established in each university. The reasons for this were mainly political. The emperor feared the fruits of enlightenment embodied in the French Revolution, and therefore was sympathetic to the Catholic restoration that joinmed issue with the spirit of freethinking which had spread over Bohemia. Bolzano, who had taken orders in 1804, was called to the new chair at the University of Prague in 1805.

Spiritually, Bolzano belonged to the Enlightenment. Both his religious and social views made him quite unsuitable for the intended task, and difficulties were inevitable. His appointment was received in Vienna with suspicion, and it was not approved until 1807.

Bolzano's lectures, in which he expounded his own views on religion, were enthusiastically received by his students; in particular, his edifying Sunday speeches (*Erbuungsreden, in Gesammelte Schriften*, I) to the students were warmly applauded. He was respected by his colleagues, and in 1815 became a member of the Königlichen Böhmischen Gesellschaft der Wissenschaften and, in 1818, dean of the Prague philosophical faculty.

In the struggle between the Catholic restoration and the Enlightenment action against Bolzano was post poned until 1816, when a charge was brought against him at the court in Vienna; his dismissal was issued on 24 December 1819. He was forbidden to publish and was put under police supervision. Bolzano repeatedly refused to recant the heresies of which he was accused, and in 1825 the action came to an end through the intervention of the influential nationalist leader J. Dobrovsky.

From 1823 on, Bolzano spent summers on the estate of his friend J. Hoffmann, near the village of Těchobuz in southern Bohemia. He lived there permanently from 1831 until the death of Mrs. Hoffmann in 1842. Then he returned to Prague, where he continued his mathematical and philosophical studies until his death.

Though Bolzano's career was concerned mainly with social, ethical, and religious questions, he was irresistibly attracted by philosophy, methodology of science, and especially mathematics and logic. His philosophical education—which acquainted him with the Greeks and with Wolff, Leibniz, and Descartes—convinced him of the necessity of forming clear concepts and of sound reasoning, starting from irreducible first principles and using only intrinsic properties of defined concepts. Such methods could not take into account properties alien to their definition, such as geometrical evidence (see *Beyträge*). On occasion he applied these principles with remarkable results; on other occasions, however, his philosophical approach, particularly to mathematics, led him to introduce insufficiently founded and incorrect assumptions. Such was the case in *Die drey Probleme*, which was explicitly intended to lead to a completely new theory of space—which, of course, it failed to do. In the domain of mathematical analysis, however, Bolzano's struggle for clear concepts did lead to profound results that, unfortunately, did not attract the attention of the mathematical world or influence the development of mathematics.

Around the turn of the nineteenth century, mathematicians in Europe were concerned with two major problems. The first was the status of Euclid's parallel postulate, and the second was the problem of providing a solid foundation for mathematical analysis, so as to remove the so-called scandal of the infinitesimals. Bolzano tried his hand at both problems, with varying success.

In 1804 he published his *Betrachtungen über einige Gegenstände der Elementargeometrie*, in which he tried to base the theory of triangles and parallels on a theory of lines, without having recourse to theorems of the plane. The full development of this theory of lines was postponed—and although Bolzano often returned to the theory of parallels (without success), his linear theory was never completed.

In the course of the following years, Bolzano became acquainted with the extensive work done in the theory of parallels, such as that of A.M. Legendre and F.K. Schweikart. There are no indications that he ever knew of the final breakthrough to non-Euclidean geometry by Nikolai Lobachevski and János Bolyai, although the latter's work was published in 1832 in Hungary. Bolzano's own manuscript "Anti-Euklid" follows a different line of thought and is devoted mainly to methodologocal criticism of Euclid's *Elements*. In fact, in his methodological principles he went so far as to require definitions of such geometrical notions as those of (simple closed) curve, surface, and dimension (see *Die drey Probleme; Ueber Haltung;* "Geometrische Begriffe" and E. Winter, *Die historische Bedeutung*), and to require proofs of such seemingly evident statements as "A simple closed curve divides the plane into two parts," which is now known as the Jordan curve theorem. Indeed, the discussion in "Anti-Euklid" confirms the opinion held by H. Hornich that Bolzano was the first to state this as a theorem (requiring proof). The problems raised in this connection by Bolzano found their final solution at the end of the nineteenth century and the beginning of the twentieth in that branch of mathematics called topology (for a discussion of these matters, see Berg, *Bolzano's Logic*).

It should be emphasized that Bolzano was not the only one, or even the first, to be concerned with the problem of rigorous proofs in mathematics. A curious fact, however, is that although many of the mathematicians actively interested in the problem of the foundation of mathematical analysis surpassed him in mathematical skill, Bolzano overcame them decisively in the foundation of analysis, in which as early as 1817 (see *Rein analytischer Beweis*) he obtained fundamental results, which were completed in 1832–1835 in his theory of real numbers (see Rychlik, *Theorie der reellen Zahlen*).

The introduction of infinitesimals by Newton and Leibniz in the seventeenth century met with violent resistance from philosophers and mathematicians, and vivid discussions on infinitesimal quantities went on throughout the eighteenth century. Bishop Berkeley's attack in *The Analyst* (1734) is well known. Although Leibniz himself did not consider the existence of infinitesimals to be well founded, and held that their use could be avoided, he admitted them as ideal quantities, which could be handled in calculations like ordinary quantities(except that they equal their finite multiples). These arithmetical properties, however, formed the weak point in the theory of infinitesimals because of the lack of an exhaustive description of the real number system, which was accomplished only in the second half of the nineteenth century. How badly the general laws of arithmetic were understood may be illustrated by the problem of division by zero. This problem kept Bolzano busy from 1815 on, and he never fully got to the bottom of it, as can be seen, for instance, in §34 of, his *Paradoxien des Unendlichen*, where he admits identities of the form $A/0 = A/0$, despite his knowledge of Ohm's solution.

To overcome the difficulties presented by infinitesimals, Lagrange proposed to base analysis on the existence of Taylor's expansion for functions, and this attitude was widely accepted for a time. Bolzano did not escape its influence, and made extensive studies on Taylor's theorem (see *Der binomische Lehrsatz* and "Miscellanea mathematica").

A different position was held by d'Alembert, who proposed to found differential calculus on the notion of limit and contended that differential calculus does not treat of infinitely small quantities, but of limits of finite quantities.

Certainly d'Alembert's opinion impressed his contemporaries, and many attempts, such as Lagrange's, were made to free differential calculus from infinitesimals. The first successful attempt was made by Bolzano in his *Rein analytischer Beweis* (1817), which is devoted to a proof of the important theorem which states that if for two continuous functions f and ϕ we have $f(\alpha) < \phi(\alpha)$ and $f(\beta) > (\beta) \geq \alpha(\beta)$, then there is an x between α and β such that $f(x) = \phi(x)$.

Bolzano argues that a sound proof of this theorem presupposes a sound definition of continuous function. In his introduction he gives such a definition, which is important because it is the first that does not involve infinitesimals, and is, essentially, the one used up to now. In the more accurate formulation of Volume **I** of the *Functionenlehre*, it reads: If $F(x+\Delta x)$-Fx in absolute value becomes less than an arbitrary given fraction $1/N$, if one takes Δx small enough, and remains so, the smaller one takes Δx, the function Fx is said to be continuous (in x). Bolzano also distinguishes between right and left continuity.

It should be noted that in 1821 Cauchy, in his *Cours d'analyse*, adopted a different definition: $f(x+\alpha) - f(x)$ infinitely small for all infinitely small α. Because of its elegance, this definition was generally accepted.

In his proof of the theorem in the *Rein analytischer Beweis*, Bolzano uses a lemma that later proved to be the cornerstone of the theory of real numbers. He was fully aware of the paramount importance of this theorem, and he formulated it with great generality, as follows: If a property *M* does not hold for all x less than a certain *u*, then there is a quantity *U*, which is the greatest of all those for which it holds that all x less than it have property *M*.

In modern terminology, *U* is the greatest lower bound of the (nonempty) set of *x* for which *M* does not hold.

Though the two theorems mentioned above already bear witness to the outstanding content of the *Rein analytischer Beweis*, it contains another theorem of equal importance, which is known as Cauchy's condition of convergence. Bolzano devotes a whole section to it and proves that if a sequence $F_1(x), F_2(x), F_3(x),\ldots, F_n(x),\ldots, F_{n+r}(x)$ is such that the difference between the *n*th term $F_n x$ and every later one $F_{n+r}(x)$ remains less than any given quantity if only n has been taken large enough, then there is a fixed quantity, and only one, to which the terms approach —as near as one likes, if one continues the sequence far enough.

The proofs of these theorems are incomplete, and were bound to be so, because complete proofs would require a precise notion of quantity (real number), which Bolzano did not have at that time. He was aware of at least some of the difficulties involved, because his methodology, as expounded in the *Beyträge*, demanded the systematic development of a theory of real numbers that should logically precede his theory of real functions.

A fairly complete theory of real functions is contained in Bolzano's *Functionenlehre*, including many of the fundamental results that were rediscovered in the second half of the nineteenth century through the work of K. Th. Weierstrass and many others.

In the first part, concerning continuous functions, it is shown that a function *Fx* which is unbounded on the closed interval [*a,b*] cannot be continuous on [*a,b*]. The proof uses the so-called Bolzano-Weierstrass theorem that a bounded infinite point set has an accumulation point. For this theorem Bolzano refers to his own work, in which up to now it has not been found.

Functions continuous on a closed interval attain there a maximal and a minimal value. Bolzano sharply distinguishes between continuity and the property of assuming intermediate values, and proves that continuous functions assume all values intermediate between any two function values, while the converse is shown not to be true.

In §13 Bolzano notices a property of continuous function which is rather close to uniform continuity, a notion which is due to E. Heine (1870, 1872). In connection with the function

Which is continuous on (0,1), he observes that though a function may be continuous on the open interval (*a,b*), it does not follow that a real number *e*, independent of x in (a,b), exists, such that one need not choose $\Delta x < e$ in order that $F(x + \Delta x) - Fx < 1/N$. Indeed, if in the example x approaches i, the x has to be taken increasingly smaller in order that $F(x + \Delta x) - Fx < 1/N$.

As K. Rychlik has pointed out in his commentary in Volume I of the *Schriften*, the said property is weaker than uniform continuity. One may be tempted, however, to assume that Bolzano intended uniform continuity and that only the formulation is defective. The more so, when in "Verbesserungen und Zusätze" we find the correct theorem: If the function Fx is continuous on the closed interval [*a,b*], then there exists a (real) number *e* such that for all x in [*a,b*] the Δx need not be chosen $<e$ in order that $F(x + \Delta x) - Fx <$ a given number $1/N$. Further reading reveals, however, that Bolzano had no clear notion of uniform continuity after all.

Careful attention is paid to the connection between monotonicity and continuity. Thereby the following correction to §79 of the *Functionenlehre* in "Verbesserungen und Zusätze" should not remain unnoticed: If the (real) function *Fx* increases (or decreases) steadily on the closed interval [a,b], then *Fx* is continuous on [*a,b*], with the exception of a set of isolated values of *x* which may be infinite or finite.

The most remarkable result of the *Functionenlehre* however, is the construction in §75 of the so-called Bolzano function. There Bolzano constructs a function as the limit of a sequence of continuous functions which is continuous on the closed interval [0,1] such that it is in no subinterval monotone. The importance of this function, however, derives from another property—its nondifferentiability.

The second part of the *Functionenlehre* is devoted to derivatives. Particular emphasis is laid on the distinction between continuity and differentiability. Bolzano shows that the above-mentioned function, though continuous in [0,1]—which is not proved—fails to be differentiable on an everywhere dense subset of [0,1]. In fact, the function is nowhere differentiable on [0,1]. This example preceded by some forty years that of Weierstrass, who in 1875 published a different example of a nowhere differentiable continuous function which roused wide interest and even indignation.

Bolzano erroneously believed that his function was continuous because it was the limit of continuous functions; in explanation it may be remarked that Cauchy made the same error. Apparently Bolzano was not aware of a counterexample given by N.H. Abel in 1826.

Though the second part of the *Functionenlehre* contains many interesting results, it contains as many errors, such as the statement that the derivative of an infinite series is the sum of the derivatives of its terms, and the conclusion that the limit of a sequence of continuous functions again is a continuous function. Both errors tie up with the notion of uniformity and therefore are explainable; the following error is less easy to understand. In 1829 Cauchy put forward the function

$C(x) = e^{-1}/x^2$ (to be completed by $C[0] = 0$)

an example of a function, different from zero for all $x \neq 0$, having all its derivatives zero for $x = 0$ and, hence, not admitting a Taylor expansion in the neighborhood of $x = 0$. Bolzano knew of this example in 1831 (see "Miscellanea mathematica," p. 1999), yet in the *Functionenlehre*, §89, we find the following theorem:

if $F^{n+r}a = 0$ for $r > 0$, then

which is clearly refuted by Cauchy's example.s

The firm base on which the theory of functions was to rest, according to Bolzano's methodology—the theory of quantities (real numbers)— was completed in 1832–1835. Like most of Bolzano's mathematical work, it remained in manuscript and was published for the first time only in 1962 (see Rychlik, *Theorie der reellen Zahlen*). As a result, this bold enterprise failed to exercise any influence on the development of mathematics, which in the second half of the nineteenth century independently took the same course.

Real numbers occur in Bolzano's writings under the name of measurable infinite number-expressions. They make their appearance in "Miscellanea mathematica," pt. 22, p. 2000–2001 (1832), in connection with the geometric progression, which has inspired many interesting ideas. The representation of the sum S of an infinite geometric progression as given in the footnote to §18 of the *paraoxien des Unendlichen* is paradigmatic.

Bolzano's idea is that descriptions of (real) numbers make sense only if they permit determination of the numbers described with an arbitrarily high degree of precision by means of rational numbers. In general, these descriptions require an infinite number of arithmetical operations to be carried out—for instance, the sum S of a geometric progression. These are the infinite number expressions with which Bolzano is concerned. If the results obtained by carrying out only a finite number of operations is always positive, the number expression is called positive.

An infinite number expression S is called measurable (or determinable by approximation) if to any natural number q there is an integer p, such that

where p_1 and p_2 are positive (infinite) number expressions.

Infinitely small numbers are those for which all $p = 0$, i.e., those S for which $S=P_1=1/q-P_2$

as well as their opposites.

An essential requirement is that measurable numbers differing in an infinitesimal number have to be considered as equal. Therefore, equality is defined by equality of p for all q in the above representation of infinite number expressions. On the basis of these definitions, Bolzano completed his systematic exposition of the theory of real numbers and, thereby, of mathematical analysis.

The elaboration is not quite satisfactory because of many errors due to his insufficient mathematical skill (for interpretations and evaluation of Bolzano's theory, see Laugwitz, "Bemerkungen";van Rootselaar, "Bolzano's Theory of Real Numbersr"; Rychlik, *Theorie der reellen Zahlen*).

The essential difference between Bolzano's incomplete theory of real numbers and those of for instance, theory Weierstrass and Georg Cantor are marked by the shift from intensional meaning, in Bolzana's work, toward a general tendency to extensionality, and, above all, by the possibility of creating new mathematical objects by means of definition by abstraction, based on equivalence relations, of which Bolzana was unaware. These difference also appear clearly in his *Paradoxien des Unendlichen*, which contains many interesting fragments of general set theory.

The existence of infinite sets is proved in a way similar to that followed by Richard Dedekind in his memoir *Was sind und was sollen die Zahlen* (1887). Most noteworthy, however, is that in §20 of, *Paradoxien des Unendlichen* Bolzano is at the border of cardinal arithmetic, a border which he is unable to cross. There he notices a property of infinite sets: that they may be brought into one-to-one correspondence with a proper subset. In fact, he observes that this will always be the case with infinite sets.

That two sets may be brought into one-to-one correspondence is no reason for him to consider them to be composed of the same number of elements (*Paradoxien des Unendlichen*, §21), however and he sees no reason to consider such sets as equal. On the contrary, in order for two sets to be considered as equal, he argues, they must be defined on the same basis (*gleiche Bestimmungsgründe haben*). Needless to say, this is too vague to be dealt with mathematically. Here again, we see that Bolzana halts at a point where application of the method of definition by abstraction would have opened entirely new fields of knowledge.

Precisely that property of infinite sets noticed by Bolzano was afterward used by Dedekind as a definition of the infinite (1882). The introduction of equivalence classes of sets under one-to-one correspondence was fully exploited by Cantor in his theory of cardinals, a very important chapter of general set theory.

Bolzano planned to elaborate the methodology begun in his *Beyträge* and to develop it into a complete theory of science, of which a treatise on logic was to form the cornerstone. From 1820 on, he worked steadily on it, and his four-volume treatise *Wissenschaftslehre* appeared in 1837. The plan of the *Wissenschaftslehre* appears clearly from the following subdivision (see Kambartel, *Bernard Bolzano's Grundlegung der Logik*, pp. 14–17):

(1) Fundamental theory: proof of the existence of abstract truths and of the human ability to judge.

(2) Elementary theory: theory of abstract ideas, propositions, true propositions, and deductions.

(3) Theory of knowledge: condition of the human faculty of judgement.

(4) Heuristics: rules to be observed in human thought in the search for truths.

(5) Proper theory of science: rules to be observed in the division of the set of truths into separate sciences and in their exposition in truly scientific treatises.

The work did not induce a complete revision of science, as Bolzano hoped, but, on the contrary, remained unnoticed and did not exercise perceptible influence on the development of logic. Some of the innovations in logic contained in the first two volumes did attract attention, as well as excessive praise—notably from Edmund Husserl and Heinrich Scholz (see Berg, *op. cit*.; Kambartel, *op. cit*.; and the literature cited in them).

The rise of logical semantics, initiated by Alfred Tarski in the 1930's has led to a revival of the study of Bolzano' logic in the light of modern logic (see berg, *op. cit*.) and of his theory of an ideal language.

The heart of Bolzana' logic is formed by his concepts of (abstract) proposition (*Satz an sich*), abstract idea (*Vorstellung an sich*), truth, and the notion of derivability (Ableitbarkeit) and entailment (Abfolge).

These notions may be explained with the help of Bolzano's example:

(a) Cajus is a human being.

(b) All human beings have immortal souls.

(c) Cajus has an immortal soul.

First of all, (*a*) expresses an abstract proposition, which in itself has no real existence, but is something to which (a) refers and which is either true of false. An abstract Proposition may be expressed in many ways linguistically, and it is said to be true if it asserts something as it actually is (*"so wie es ist," Wissenschaftslehre, §25*).

Bolzano argues that any proposition may be expressed in the normal from "*A* has *b*." For instance, (a′)Cajus has human existence is the normal form of the Proposition expressed by (*a*).

Parts of Propositions which are not themselves Propositions are (abstract) ideas; for example, in (a′) the expression "human existence" refers to an abstract idea.

Between abstract propositions there exist relations, among which those of consistency and derivability are of paramount importance. Propositions $A, B, C,\ldots$ are called consistent with respect to the common ideas $i, j,\ldots$ turn the propositions $A, B, C, \ldots$ into simultaneously true propositions $A', B', C', \ldots$. Propositions $A', B', C', \ldots$ are called derivable from $A, B, C, \ldots$ with respect to the ideas i, j, whenever $A, B, C, \ldots, A', B', C', \ldots$ are consistent with respect to $i, j,\ldots$ and if any substitution i', j', for i, j, that turns $A', B', C', \ldots$ into true Propositions also turns $A', B', C', \ldots$ into true propositions. According to Bolzano, (c) is derivable from (*a*) and (*b*).

The relation of entailment (*Abfolge*) may subsist between true Propositions, and refers to the situation that A is true because A_1, A_2, … are true. The treatment of this notion, however, is rather unsatisfactory (see Berg, *op. cit*,; Buhl, *Ableitbarkeit und Abfolge;* Kambartel, *op.cit.* for details).

The resemblance that many of the concepts introduced by Bolzano bear to modern logic has led to the opinion that Bolzano may be considered a true Precursor of modern logic. (For a detailed account, consult Berg, *op. cit*.; and Kambartel, *op. cit*.; for Bolzano—s philosophy,*I* Fujita, Borutsāno no tetsugaku [“Bolzano’s Philosophy”]).

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B. Van Rootselaar