# George Boole | Encyclopedia.com

Complete Dictionary of Scientific Biography COPYRIGHT 2008 Charles Scribner's Sons
22-28 minutes

---

(*b*. Lincoln, England, 1815; *d*. Cork, Ireland, 1864)

*mathematics*.

George Boole was the son of John Boole, a cobbler whose chief interests lay in mathematics and the making of optical instruments, in which his son learned to assist at an early age. The father was not a good businessman, however, and the decline in his business had a serious effect on his son's future. The boy went to an elementary school and for a short time to a commercial school, but beyond this he educated himself, encouraged in mathematics by his father and helped in learning Latin by William Brooke, the proprietor of a large and scholarly circulating library. He acquired a knowledge of Greek, French, and German by his own efforts, and showed some promise as a classical scholar; a translation in verse of Meleager's "Ode to the Spring" was printed in a local paper and drew comments on the precocity of a boy of fourteen. He seems to have thought of taking holy ordres, but at the age of fifteen he began teaching, soon setting up a school of his own in Lincoln.

In 1834 the Mechanics Institution was founded in Lincoln, and the president, a local squire, passed Royal Society publications on to institution's reading room, of which John Boole became curator. George, who now devoted his scanty leisure to the study of mathematics, had access to the reading room, and grappled, almost unaided, with Newton's *Principia* and Lagrange's *Mécanique analytique*, gaining such a local reputation that at the age of nineteen he was asked to give an address on Newton to mark the presentation of a bust of Newton, also a Lincolnshire man, to the Institution. This address, printed in 1835, was Boole's first scientific publication. In 1840 he began to contribute to the recently founded *Cambridge Mathematical Journal* and also to the Royal Society, which awarded him a Royal Medal in 1844 for his papers on operators in analysis; he was elected a fellow of the Royal Society in 1857.

In 1849, Boole, on the advice of friends, applied for the professorship of mathematics in the newly established Queen's College, Cork, and was appointed in spite of his not holding any university degree. At Cork, although his teaching load was heavy, he found more time and facilities for research. In 1855 he married Mary Everest, the niece of a professor of Greek in Queen's College and of Sir George Everest, after whom Mount Everest was named.

Boole was a clear and conscientious teacher, as his textbooks show. In 1864 his health began to fail, and his concern for his students may have hastened his death, since he walked through rain to a class and lectured in wet clothes, which led to a fatal illness.

Boole's scientific writings consist of some fifty papers, two textbooks, and two volumes dealing with mathematical logic. The two textbooks, on differential equations (1859) and finite differences (1860), remained in use in the United Kingdom until the end of the century. They contain much of Boole's original work, reproducing and extending material published in his research papers. In the former book, so much use is made of the differential operator *D* that the method is often referred to as Boole's, although it is in fact much older than Boole. Both books exhibit a great technical skill in the handling of operators: in the volume on finite differences, an account is given of the operators $\pi$ and $\varrho$, first introduced in Boole's Royal Society papers. The basic operators of this calculus, $\Delta$ and $E$, defined by his equations

$$\Delta u_x = u_{x+1} - u_x,\ E\, u_x = u_{x+1};$$

Boole then defines his new operators by the operational equations

$$\pi = x\Delta,\ \varrho = xE,$$

and shows how they can be used to solve certain types of linear difference equations with coefficients depending on the independent variable. These operators have since been generalized by L.M. Milne-Thomson.

In papers in the *Cambridge Mathematical Journal* in 1841 and 1843, Boole dealt with linear transformations. He showed that if the linear transformation

$$x = pX + qY,\ y = rX + sY$$

is applied to the binary quadratic form

$ax^2 + 2hxy + by^2$

to yield the binary quadratic form

$AX^2 + 2HXY + BY^2$,

then $AB - H^2 = (ps - qr)^2 (ab - h^2)$.

The algebraic fact had been partly perceived by Lagrange and by Gauss, but Boole's argument drew attention to the (relative) invariance of the discrimiant $ab - h^2$, and also to the absolute invariants of the transformation. This was the starting point of the theory of invariants, so rapidly and extensively developed in the second half of the nineteenth century; Boole himself, however, took no part in this development.

Other papers dealt with differential equations, and the majority of those published after 1850 studied the theory of probability, closely connected with Boole's work on mathematical logic. In all his writings, Boole exhibited considerable technical skill, but his facility in dealing with symbolic operators did not delude him into an undue reliance on analogy, a fault of the contemporary British school of symbolic analysis. E.H. Neville has remarked that mathematicians of that school treated operators with the most reckless disrespect, and in consequence could solve problems beyond the power not merely of their predecessors at the beginning of the century but of their inhibited successors at the end of the century, obtaining many remarkable and frequently correct formulas but ignoring conditions of validity.

Boole greatly increased the power of the operational calculus, but seldom allowed himself to be carried away by technical success: at a time when the need for precise and unambiguous definitions was often ignored, he was striving, although perhaps not always with complete success, to make his foundations secure. There is a clear and explicit, although later, statement of his position in his *Investigation of the Laws of Thought;* there are, he says, two indispensable condition for the conditions for the employment of symbolic operators: "First, that from the sense once conventionally established, we never, in the same process of reasoning, depart; secondly, that the laws by which the process is conducted be founded exclusively upon the above fixed sense or meaning of the symbols employed. "With the technical skill and the desire for logical precision there is also the beginning of the recognition of the nonnumerical variable as a genuine part of mathematics. The development of this notion in Boole's later and most important work appears to have been stimulated almost accidentally by a logical controversy.

Sir William Hamilton, the Scottish philosopher (not to be confused with the Irish mathematician Sir William Rowan Hamilton), picked a logical quarrel with Boole's friend Augustus De Morgan, the acute and high-minded professor of mathematics at University College, London. De Morgan's serious, significant contributions to logic were derided by Hamilton, on the grounds that the study of mathematics was both dangerous and the useless-no mathematician could contribute anything of importance to the superior domain of logic. Boole, in the preface to his *Mathematical Analysis of Logic* (1847), demonstrated that, on Hamilton's own principles, logic would form no part of philosophy. He asserted that in a true classification, logic should not be associated with metaphysics, but with mathematics. He then offered his essay as a construction, in symbolic terms, of logic as a doctrine, like geometry, resting upon a groundwork of acceptable axioms.

The reduction of Aristotelian logic to an algebraic calculus had been more than once attempted; Leibniz had produced a scheme of some promise. If the proposition "ALL *A* is *B*" is written in the form *A*/*B*, and "All *B* is *C*" in the form *B*/*C*, then it is tempting to remove the common factor *B* from numerator and denominator and arrive at *A*/*C*, to be correctly interpreted as the conclusion "All *A* is *C*." Any attempt to extend his triviality encountered difficulties: Boole's predecessors had tried to force the algebra of real numbers onto logic, and since they had not envisaged a plurality of algebras, it was believed that only if the elementary properties of the symbols implied formal rules identical with those of the algebra of real numbers could the subject be regarded as a valid part of mathematics. Boole recognized that he had created a new branch of mathematics, but it is not clear whether he appreciated that he had devised a new algebra. He appears not to have known that geometries other than Euclidean could be constructed; but he knew of Rowan Hamilton's quaternion, an algebra of quadruplets in which products are noncommutative, for one of his minor papers (1848) deals with some quaternion matters. Grassmann's similar, if more general, work in the *Ausdehnungslehre* (1844) seems to have been unknown. Boole, then, knew of an algebra similar to, but not identical with, the algebra of real numbers.

If we consider a set *U*, the universal set or the universe of discourse, often denoted by 1 in Boole's work, subsets can be specified by elective operators *x, y, …*, so that *xU* is the subset of *U* whose elements have the property defining the operator x. Thus, if *U* is the set of inhabitants of New York, we can select those who are, say, male by an elective operator *x* and denote the set of male inhabitants of New York by *xU*. Similarly, the left-handed inhabitants of New York may be denoted by *yU*, and blue-eyed inhabitants by *zU*, and so on. The elective operators may be applied successively. Thus we may first select all the males and from these all the lefthanders by the symbolism *y(xU);* if we first select all the left-handers and from these all the males, we have the symbolism *x(yU)*. Since in each case the final set is the same, that of all left-handed males, we can write *y(xU)=x(yU)*, or, since the universe of discourse *U* is understood throughout, simply write *yx=xy*. The analogy with the commutative algebraic product is clear. The associative law for products, *x()yz=(xy)z*, can be verified at once in this interpretation, since each side denotes the set of those who are at once male, left-handed, and blue-eyed; Boole uses this without bothering to give any explicit justification. He was careful, however, to remark that although an analogy exists, the evidence on which the laws are based in his work is not related to the evidence on which the laws of the algebra of real numbers are based. To select the set of males from the set of males is merely to arrive at the set of males; thus the definition of the operator $x$ leads to the idempotent law $x(xU)= xU$, or $x^2=x$, the first break with ordinary algebra.

The Product or intersection operation can also be regarded as a symbolic expression of the logical concept of conjunction by means of the conjunctive "and," since $xy$ will denote the set of those inhabitants of New York who are at once male *and* left-handed.

If $xU$ is the subset of males in the universal set $U$, it is natural to write the set of nonmales, that which remains when the set of males is subtracted from $U$, as $U — xU$, or, briefly, $1 — x$. This set, the complement of $x$ relative to $U$, which Boole for brevity denoted by $\bar{x}$, can be regarded as arising from the application of the logical negation "not" to the set x. Addition has not yet been defined, but Boole did not hesitate to rewrite the equation $\bar{x}=1— x$ in the form $x + \bar{x}= 1$, implying that the universal set is made up of the elements of the subset x or of the subset not-x; this suggests that the sign + is the symbol for the connective "or." But the word "or" in English usage has an inclusive and an exclusive sense:"either…or…and possibly both" and "either…or…but not both." Boole chose the exclusive sense, and so did not allow the symbolism x+y unless the sets x, y were mutually exclusive.

Modern usage takes x + y for the union or logical sum, the set of elements belonging to at least one of x, y: this union Boole included in his symbolism as $x + x\bar{y}$. Kneale suggests that Boole' choice of the exclusive sense for the symbol + was caused by a desire to use the minus sign (-) as the inverse of the plus the inverse sign (+). If y is contained in x, x-y can consistently denote those elements of x which are not elements of y —the complement of relative to x —but if + is used in the inclusive sense, then the equations $x = y + z$, $x = y + w$ do not imply $z = w$, so that $x - y$ is essentially indeterminate. Alternately, a use of the idempotent law implies that

$$(x - y)^2=x - y,$$

and a further application of this law suggests that from

$$x^2 - 2xy + y^2 = x - y$$

it follows that

$$x - 2xy + y = x - y$$

and, hence, that $y = xy$; this is a symbolic statement that y is a subset of x. Boole was thus led to the use of the sign + in the exclusive sense, with the sign — as its inverse.

The idempotent law $x^2 = x$ is expressed in the form $x(1 – x) = 0$, but it is not altogether clear whether Boole regarded this as a deduction or as a formulation of the fundamental Aristotelian principle that a proposition cannot be simultaneously true and false. Some of the obscurity is due to the fact that Boole does not always make clear whether he is dealing with sets, or with propositions, or with an abstract calculus of which sets and propositions are representations.

Much of the 1847 tract on the mathematical analysis of logic is devoted to symbolic expressions for the forms of the classical Aristotelian propositions and the moods of the syllogism. The universal propositions "All *X*'s are *Y*'s" "No *X*'s are *Y*'s take the forms $x(1 - y) = 0$, $xy = 0$. The Particular propositions "Some *X*'s are *Y*'s" "Some *X*'s" do not take what might appear to be the natural forms $xy \neq 0$, $x(1 - y) \neq 0$, possibly because Boole wished to avoid inequalities and to work entirely in terms of equations. He therefore introduced an elective symbol, $v$ any elements common to $x$ and $y$ constitute a subset $v$ which, he says, is "indefinite in every respect but this" — that it has some members. The two particular propositions he wrote in wrote in the forms $xy = v$, This ill-defined symbol needs careful handling when the moods and figures of the syllogism are discussed. Thus the premises "All *Y*'s are *X*'s" "No *Z*'s are *Y*'s give the equations $y = vx$, $0 = zy$, with the inference $0= vzx$ to be interpreted as "SomeX' and — $v$ is regarded as the representation of some only with respect to the class X."

A similar obscurity is encountered when an attempt is made to define division. If $z = xy$, what inferences can be drawn about $x$, in the hope of defining the quotient $z/y$? Since $z$ is $y$, $yz = z$; thus $x$, which contains $z$, contains $yz$. Any other element of $x$ that is not in $z$ cannot be in $y$, and hence $x$ is made up of $yz$ and an indeterminate set of which all that can be said is that its elements belong neither to $y$ nor to $z$, and thus belong to the intersection of $1— y$ and $1 – z$. Thus

$$z/y = yz + \text{an indefinite portion of } (1- y)\ (1 - z)$$

Boole gave this result as a special case of his general expansion formula, and his argument is typical of that used to establish the general theorem. From $y+\bar{y} = 1$, $z+\bar{z}= 1$, it follows that is, the universe of discorse is the sum of the subsets. Hence, any subset whatsoever will be at most a sum of elements from each of these four subsets; thus

$$z/y = Ayz + By\bar{z}+ Cy\bar{z}+ Dy\bar{z},$$

with coefficients A, B, C, D to be determined. First, set $y = 1, z = 1$, so that $\bar{y} = \bar{z} = 0$; then $A = 1$. Next, set $y = 1, z = 0$, so that $\bar{y} = 0, \bar{z} = 1$; then $B = 0$. Third, set $y = 0, z = 1$, so that $\bar{y} = 1, \bar{z} = 0$; if the term in yz were present, then C would have to be infinite; hence, the term in $\bar{y}z$ cannot appear. Finally, if $y = z = 0$, the coefficient $D$ is of the form 0/0, which is indeterminate. This asserts the possible presence of an indefinite portion of the set $\bar{y}\bar{z}$. Thus, as before,

$z/y = yz$ + an indefinite portion of $y\bar{z}$,

or, as Boole frequently wrote it,

Schröeder showed that the introduction of division is unnecessary. But the concept of the "development" of a function of the elective symbols is fundamental to Boole's logical operations and occupies a prominent place in his great work on mathematical logic, the *Investigation of the Laws of Thought*. If $f(x)$ involves $x$ and the algebraic signs, then it must denote a subset of the universe of discourse and must therefore be made up of elements from $x$ and $\bar{x}$. Thus

$f(x) = Ax + B\bar{x}$,

where the coefficients $A$ and $B$ are determined by giving $x$ the values of 0 and 1. Thus

$f(x) = f(1)x + f(0)(1 - x)$,

which in the *Mathematical Analysis of logic* Boole regards as a special case of MacLaurin's theorem, although he dropped this analogy in the *Investigation of the Laws of Thought*. A repeated application of this method to an expression $f(x, y)$ containing two elective symbols yields

$f(x, y)= f(1, 1)xy + f(1, 0)x(1-y)$

$+f(0, 1)(1-x)y + f(0, 0)(1-x)(1-y)$,

and more general formulas can be written down by induction. Logical problems which can be expressed in terms of elective symbols may then be reduced to standard forms expediting their solution.

Boole's logical calculus is not a two-valued algebra, although the distinction is not always clearly drawn in his own work. The principles of his calculus, as a calculus of sets, are nowhere set out by him in a formal table, but are assumed, sometimes implicitly, and are, save one, analogous to the algebraic rules governing real numbers:

$xy = yx$

$x + y = y + x$

$x(y+z) = xy + xz$

$x(y-z) = xy - xz$.

If $x = y$, then

$xz = yz$

$x + z = y + z$

$x - z = y - z$.

$x(1 - x) = 0$.

Of these, only the last has no analogue in the algebra of real numbers. These principles suffice for the calculus of sets. But Boole observes that in algebra the last principle is an equation whose only roots are $x = 0, x = 1$. In the calculus of sets this would assert that any set is either the null set or the universal set. Boole added this numerical interpretation in order to establish a two-valued algebra, of which one representation would be a calculus of propositions in which the truth of a proposition $X$ is denoted by $x = 1$ and its falsehood by $x = 0$: the truth-value of a conjunction "$X$ and $Y$" will be given by $xy$, and of an exclusive disjunction "$X$ or $Y$" by $x + y$. The distinction between propositions and propositional fonctions, not drawn by Boole, was made later by C. S. Peirce and Schröder.

The use of $x + y$ to denote the exclusive sense of "or' led to difficulties, such as the impossibility of interpreting $1 + x$ and $x + x$, which Boole surmounted with considerable ingenuity. But Jevons, in his *Pure Logic* (1864), used the plus sign in its inclusive (and/or) sense, a use followed by Venn and C. S. Peirce and since then generally adopted. Peirce and Schroder emphasized that the inclusive interpretation permits a duality between sum and product, and they also showed that the concepts of subtraction and division are superfluous and can be discarded. With the use of $x + y$ to denote "either $x$ or $y$ or both," the expression $x + x$ presents no difficulty, being just $x$, while $1 + x$ is the universal set 1. The duality of the two operations of sum and product exemplified by the equations $xx = x$, $x + x = x$ can now be caried further: the formulas

$xy + xz = x(y + z)$, $(x + y)(x + z) = x + yz$

are duals, since one can be derived from the other by an interchange of sum with product. This duality is clearer if these operations are denoted by the special symbols ⋂ ⋃now in general use for product and sum, that is, for intersection and union. In this notation, the preceding equations are written

$(x \cup y) \cap (x \cup z) = x \cup (y \cap z)$,

$(x \cap y) \cup (x \cap z) = x \cap (y \cup z)$.

With the inclusive interpretation, the system can now be shown to obey the dual rules of De Morgan:

In the *Investigation of the Laws of Thought*, the calculus is applied to the theory of probability. If $P(X) = x$ is the probability of an event $X$, then if events $X, Y$ are independent, $P(X \text{ and } Y) = xy$, while if $X$ and $Y$ are mutually exclusive, $P(X \text{ or } Y) = x + y$. The principles laid down above are satisfied, except for the additional numerical principal in which the allowable values of $x$ are 0 and 1, which is not satisfied. A clear and precise symbolism enabled Boole to detect and correct flaws in earlier work on probability theory.

E. V. Huntington in 1904 gave a set of independent axioms on which Boole's apparatus can be constructed, and various equivalent sets have been exhibited. One formulation postulates two binary operations (union and intersection) which have the commutative and distributive properties:

$x \cup y = y \cup x$, $x \cap y = y \cap x$

$x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$

$x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$

further, there are two distinct elements, 0 and 1, such that for all $x$

$x \cup 0 = x$, $x \cap 1 = x$;

also, for any x, there is an element $x$ (the complement) for which

The system so defined is self-dual, since the set of axioms remains unchanged if and are interchanged when 0, 1 are also interchanged. The associative laws for union and intersection are not required as axioms, since they can be deduced from the given set.

If intersection and complement are taken as the basic operations, with the associative law $x \cap (y \cap z) = (x \cap y) \cap z$ now an axiom and the relation between the basic operations given by the statements

then union can now be defined in terms of intersection and complement by the equation

0 can be defined as $x \cap \bar{x}$ and 1 as the complement of 0. The two systems are then equivalent.

The theory of lattices may be regarded as a generalization. A lattice is a system with operations ⋃, ⋂ having the commutative, distributive, and associative properties. Thus every Boolean algebra is a lattice; the converse is not true. The lattice concept is wider than the Boolean, and embraces interpretations for which Boolean algebra is not appropriate.

Boole's two-valued algebra has recently been applied to the design of electric circuits containing simple switches, relays, and control elements. In particular, it has a wide field of application in the design of high-speed computers using the binary system of digital numeration.

# BIBLIOGRAPHY

I. Original Works. Boole's papers include "Researches on the Theory of Analytical Transformations, With a Special Application to the Reduction of the General Equation of the Second Order," in *Cambridge Mathematical Journal*, **2** (1841), 64–73; "On a General Method in Analysis," in *philosophical Transactions of the Royal Society of London*, **134** (1844), 225–282. *An Address on the Genius and Discoveries of Sir Isaac Newton* was published in Lincoln in 1835.

His textbooks are *Treatise on Differential Equations* (Cambridge, 1859, and later editions); a posthumous *Supplementary Volume* (Cambridge, 1865), compiled from Boole's notes by Isaac Todhunter, and containing a list of Boole's publications; *Treatise on the calculus of Finite Differences* (Cambridge, 1860, and later editions).

On mathematical logic: *The Mathematical Analysis of Logic, Being an Essay Towards a calculus of Deductive Reasoning* (Cambridge, 1847; repr. Oxford, 1948, and in Boole's *Collected Logical Works*, I, Chicago-London, 1916); *An Investigation of the Laws of Thought, on Which Are Founded the Mathematical Theories of Logic and Probability*(London, 1854; repr. New York, 1951, and in Boole's *Collected Logical Works*, II, Chicago-London, 1916).

II. Secondary Literature. E. V. Huntington, "Sets of Independent Postulates for the Algebra of Logic," in *Transactions of the American Mathematical Society*, **5** (1904), 208–309; E. V. Huntington, "Postulates for the Algebra of Logic," in *Transactions of the American Mathematical Society*, **35** (1933), 274–304; W. Kneale, "Boole and the Revival of Logic," in *Mind*, **57** (1948), 149–175, which contains a useful bibliography; W. Kneale, "Boole and the Algebra of Logic," in *Notes and Records of the Royal Society of London*, **12** (1956), 53–63; Sir Geoffrey Taylor, "George Boole, 1815–1864," *ibid*., 44–52, which gives an account of Boole's life by his grandson.

T. A. A. Broadbent