On the origin of the Fibonacci Sequence

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Abstract

Herein we investigate the historical origins of the Fibonacci numbers. After emphasising the importance of these numbers, we examine a standard conjecture concerning their origin only to demonstrate that it is not supported by historical chronology. Based on more recent findings, we propose instead an alternative conjecture through a close examination of the historical and historical/mathematical circumstances surrounding Leonardo Fibonacci and relate these circumstances to themes in medieval and ancient history. Cultural implications and historical threads of our conjecture are also examined in this light.

Keywords: Fibonacci, Medieval, Islam (Medieval), Greek, Egyptian, Amazigh (Kabyle), Bejaia
01A13, 01A35, 01A20, 01A30

1. Introduction

Conventional wisdom suggests that the Fibonacci numbers were first introduced in 1202 by Leonardo of Pisa, better known today as Fibonacci, in his book *Liber abaci*, the most influential text on mathematics produced in Europe at that time. The Fibonacci number sequence appeared in the solution to the following problem :

> "A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each

pair begets a new pair which from the second month on becomes productive ? ”

The resulting sequence is

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \cdots \tag{1}$$

(Fibonacci omitted the first term in *Liber abaci*). The recurrence formula for these numbers is :

$$\begin{aligned} F(0) &= 0 \\ F(1) &= 1 \\ F(n) &= F(n-1) + F(n-2) \quad n > 1 \ . \end{aligned} \tag{2}$$

Although Fibonacci only gave the sequence, he obviously knew that the n^{th} number of his sequence was the sum of the two previous numbers.
Johannes Kepler, known today for the “Kepler Laws” of celestial mechanics, noticed that the ratio of consecutive Fibonacci numbers, as in for example, the ratio of the last two numbers of (1), approaches ϕ which is called the *Golden* or divine ratio (e.g. see [Cook,1979]) :

$$\frac{55}{34} = 1.618 \approx \phi = \frac{1+\sqrt{5}}{2} \tag{3}$$

In a detailed analysis [Jung&Pauli,1952/2012], the Nobel-prize winning Physicist W. Pauli discussed possible influences on Kepler concerning the formulation of his “Kepler laws”, the mathematical relations put forward by Kepler (1571-1630) in an effort to fit the astronomical data of Tycho Brahe of Copenhagen. Kepler’s laws were eventually derived by Newton through the application of Galileo’s findings in dynamics. This successful effort gave birth to the science of “Classical Mechanics”, upon which Modern Physics and all its far-reaching technological applications and philosophical concepts are based.

According to Pauli, two of the most important pervasive influences on Kepler’s beliefs originated from Pythagorean Mathematics and the Fibonacci numbers as manifested in the morphology of plants [Jung&Pauli,1952/2012, p.163,189]. In particular, Kepler’s firm belief that the number 3 (a Fibonacci number) was more important than the number 4 (which is not in the Fibonacci sequence) in contradistinction to what other competing astrologers believed. According to Dampier [Dampier,1966, p.127], “Kepler was searching . . . for the mathematical harmonies in the mind of the Creator”.

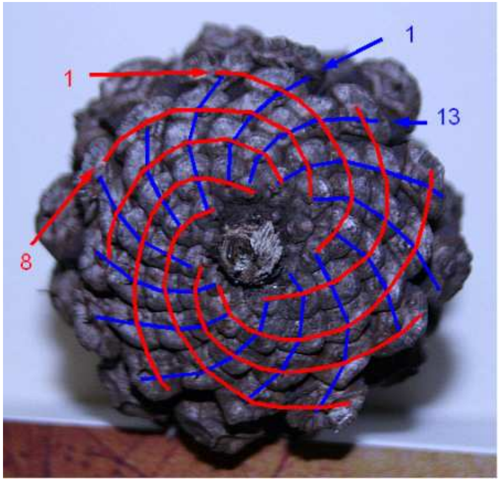

Fig. 1. Fibonacci spirals in pine cones. Courtesy of Warren Wilson College Physics Dept. [Collins,2011].

It has been noticed that leaf arrangements on certain plants and petals on some flowers follow patterns described by the Fibonacci sequence [Cook,1979, V]. The presence of Fibonacci numbers in pine cones has received particular attention [Cook,1979, VII]. From a top view of a pine cone as shown in Figure 1, two sets of spirals may be distinguished : one in the clock-wise direction and another in the counter-clockwise direction. The ratio of the numbers of each set is almost always a ratio of two Fibonacci numbers. Fibonacci helices, based on small Fibonacci numbers, appear in the arrangement of leaves of many plants on the stem. The Fibonacci spiral, also related to the Fibonacci sequence, occurs in Nature as the shape of snail shells and some sea shells. Cook [Cook,1979] found that the spiral or helix may lie at the core of life's principles : that of growth. The spiral is fundamental to organic life ranging from plants, shells to animal's horns [Cook,1979, XII]; to the periodicity of atomic elements; to microscopic DNA (the double helix) and to galaxy formations like the Andromeda nebula [Cook,1979, XX] . What is unusual is that although the rabbit model problem seems contrived and artificial i.e. rabbits *do not* reproduce in male-female twins [1], the Fibonacci numbers have universal applications and appear to be ubiquitous to nature (see for example [Stewart,1995,157-166]).

The wealth of examples cited in the previous paragraph indicates that the Fibonacci numbers represent a fundamental mathematical structure. The presence of these numbers and the Golden ratio in nature is certainly a fascinating prevalent tendency, particularly in the botanical and zoological realms [Stevens,1979, Stewart,1999]. The presence of Fibonacci num-

1. Rabbits reproduce according to litters whose size varies according to their environment. In general, they reproduce at a very high geometrical rate, that is, "like rabbits" (if amusingly but accurately put).

bers has been noted within ladder and cascade electronic network analysis [Arkin,1965,3,139-142], Modern Music [Lowman,1971,9-4,423-426&436, Lowman,1971,9-5,527-528&536-537], tributary patterns of stream and drainage patterns [Sharp,1972,10-6,643-655], Atomic Physics [Wlodarski,1963,1-4,61-63], Education [Curl,1968,6-4,266-274] and Economics [Falconbridge,1964,2,320-322]. It is not always clear why these numbers appear but in a number of instances, but they do reflect minimization or optimization principles of some sort, namely the notion that nature is efficient yet "lazy", making the most of available resources. The ubiquitous nature of Fibonacci numbers has even inspired the creation of a journal, the *Fibonacci Quarterly*.

The Golden ratio has also been used in architecture and art. It is present in many designs, from the ancient Parthenon [Cook,1979] in Athens to Stradivari's violins [Arnold,1983]. It was known to artists such as Leonardo da Vinci [Cook,1979] and musicians and composers : Bach [Norden,1964,2,219-222], Bartók [Lendvai,1971] and Debussy [Howat,1983].

What has puzzled scholars over the years is the contrast between the fundamental importance of the Fibonacci numbers themselves as opposed to the artificiality of the rabbit reproductive model by which they were apparently first introduced. Fibonacci himself does not seem to have associated that much importance to them ; the rabbit problem seemed to be a minor exercise within his work. These numbers did not assume major importance and recognition until the 19th century thanks to the work of the French mathematician Edouard Lucas.

Historians have pondered over this and doubted or wondered about the true inspiration behind these numbers and Fibonacci's knowledge of them. By his own admission, Fibonacci was influenced by Islamic scholarship (in a period of apex during Fibonacci's time). Historians have tried to assess this influence especially since Fibonacci's contributions resemble the results of Muslim scholars, in particular the work of Al-Khwârizmî (780-850 CE) (e.g. see [Zahoor,2000, CHS,1971]), a Muslim scholar who had written a book on the Hindu-Arabic numbers and from whose works words "algebra" and "algorithm" (a step-by-step procedure by which to formulate and accomplish a particular task) are derived. This has motivated historians to associate the origin of the Fibonacci sequence with Muslim scholarship in the middle ages.

The intent of this article is to offer a plausible conjecture as to the origin of the Fibonacci numbers. We start by mentioning a relatively popular conjecture and state the reasons, both mathematical and historical, which are supported from recent work (notably Roshdi Rashed [Rashed,1994,2]) as to why we believe this conjecture to be improbable. We then present our own conjecture that fits the facts as we know them today. Finally, we examine the historical implications of this conjecture concerning Fibonacci's environment after 1200 CE, in particular, the court of Frederick II (1196-1250 CE), ruler of the Holy Roman Empire.

2. The "Standard" Conjecture

2.1. Pascal's Triangle

First we generate Pascal's triangle (e.g. see [Decker&Hirshfield,1992]). This is accomplished by expanding terms in $(x+1)^m$ for $m = 0, 1, 2, \ldots$:

$$\begin{array}{cc} (x+1)^0 & 1 \\ (x+1)^1 & 1+x \\ (x+1)^2 & 1+2\,x+x^2 \\ (x+1)^3 & 1+3\,x+3\,x^2+x^3 \\ \vdots & \vdots \end{array} \tag{4}$$

and arranging the coefficients to form the following triangle :

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array} \tag{5}$$

The entries in this triangle are the binomial coefficients :

$$\left(\begin{array}{c} m \\ k \end{array} \right) = \frac{m!}{(m-k)!\ k!} \tag{6}$$

These can be related to the Pascal triangle once the latter (5) is re-arranged in a "flush-left" matrix form as shown in Table 1, where row and column numbering starts at zero. Each entry at the m^{th} row and k^{th} column of Table 1 is given by the binomial formula in (6). As shown in Figure 2, when the entries in Pascal's triangle are summed following a diagonal, the resulting sums produce exactly the Fibonacci sequence. Algebraically, we can write this diagonal sum as :

$$F(n) = \sum_{k=0}^{n-1} \left(\begin{array}{c} n-k-1 \\ k \end{array} \right) \quad n > 0 \tag{7}$$

Although Blaise Pascal(1623-1662) is credited for inventing this triangle, this was in fact known to the Chinese at least 500 years earlier [Burton,1985]. It is believed that the great Persian mathematician, philosopher and poet Omar Al-Khayyâm (1048-1131 CE) (e.g. see [CHS,1971, Coolidge,1990]) knew about binomial coefficients and this triangle. The "standard" conjecture assumes that through his contacts with the Muslims, Fibonacci would have become aware of the Chinese triangle through the work of Al-Khayyâm and from there would have realized the pattern leading to the Fibonacci numbers. In particular, Al-Karaji (also called al-Karkhi) apparently also knew about this triangle as well as binomial sums [OConner&Robertson,1999] making it plausible that Leonardo Fibonacci could have learned about this triangle and what was needed to infer the Fibonacci sequence through his Muslim contacts.

Table 1
Pascal triangle in Matrix form

		column 0	1	2	3	4
	0	1				
r	1	1	1			
o	2	1	2	1		
w	3	1	3	3	1	
	4	1	4	6	4	1

2.2. Issues with the "Standard" Conjecture

Despite the clear mathematical relationship between the Fibonacci sequence and the Pascal (or Chinese) triangle, there are major issues concerning the "Standard" conjecture, both historical and mathematical.

2.2.1. Historical Issues

According to Rashed [Rashed,1994,2, p.148], in the examination of a set of some 90 algebraic problems within *Liber abaci*, 22 of them were found to have been borrowed from Al-Khwârizmî's book on algebra (*Al-jabr wa'l muqabala*, written around 830 CE) and 53 from Abû-Kâmil's book on algebra (*Kitab fi al-jabr wa'l-muqabala*, or *Book on completion and balancing*, written around 912 CE). One deals with exactly the *same* problems with occasionally a minor change in the numerical coefficients. The borrowing is undeniably *massive*, especially in the case of Abû-Kâmil. The remaining problems in Fibonacci's *Liber Abaci* of approximately 25 in number, whose origins have not been identified, follow the models conceived by Al-Khwârizmî and Abû-Kâmil.

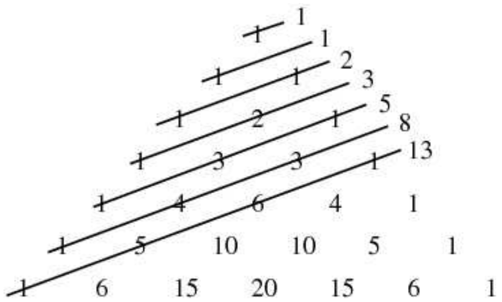

Fig. 2. Fibonacci numbers and Pascal Triangle. Courtesy of MathWorld [Mathworld].

Not only does Fibonacci owe tribute to Al-Khwârizmî and Abû-Kâmil, he remains contemporary to them, that is current on Muslim mathematics of the 9th and 10th centuries, even though Fibonacci himself lived in a later period around 1200 CE. For example, in considering the equation from Al-Khayyâm (1048-1131 CE)

$$x^3 + 2\,x^2 + 10\,x \;=\; 20 \qquad (8)$$

which was posed as a challenge to Fibonacci by John of Palermo, a problem which Al-Khayyâm could solve exactly using his devised methods of algebraic geometry, Fibonacci accepts the challenge and offers a solution in a different work called *Flos* (meaning "The flower" - a collection of solutions to problems posed in the presence of Frederick II written in 1225 CE). However, he only succeeds in obtaining a numerical approximation. There are other considerations [Rashed,1994,2, p.150] to indicate that although Fibonacci was ahead of European mathematicians, he was not current with respect to the work of Muslim mathematicians of his time. Rather, it seems that Fibonacci probably relied on translations of the works of Al-Khwârizmî by Gerard of Cremona (1114-1187 CE), the latter being a pioneer in a major effort based in Toledo, Spain to translate works written in Arabic into Latin for (Christian) Europe. We note in passing that this has prompted scholars to conjecture that there had to be a translation of Abû-Kâmil's work into Latin by the 12th century [Lévy,2000,56,58]. Consequently, Fibonacci was probably not aware of the Chinese triangle and thus the pattern leading to the Fibonacci numbers. through the work of Omar Al-Khayyam. Before one makes the accusation of plagiarism [2], which Rashed himself does not do, allowances can be made in consideration of the times that Fibonacci lived in. Fibonacci's period of time falls within the period of the crusades and this is an era of superstition and religious conflict between the Muslims and the Christian empires. Both Spain and the Holy Land were regions of military skirmishes.

Fibonacci's period precedes the rigorous scientific principles formulated by Newton and others : the era of science as we know it did not yet exist. Rather science and "non-scientific" notions exist side-by-side. For example, the activities of what we now call astronomy (a respectable science involving classical celestial mechanics) and those of astrology (a form of "divination" not taken seriously by most Western thinkers) are mingled together as part of the same activity and handled by the same scholars [3]. The "astrologers" of that time used astronomy to make predictions of planetary orbits and once these were calculated, they would make their astrological "predictions". A similar feature also applies to alchemy, the ancestor of chemistry, as well as medicine. In Fibonacci's day, one witnesses a "prehistory" to science rather than science itself. scientific fact coexists with misinformation, superstition and religious beliefs. Activities related to algebra, alchemy and astrology all represent forms of "magic" to the majority of the population at that time and face suspicion and resistance. From the Muslim side, there was an understandable resentment of having their great scholarly works copied or "plagiarized" by either Jewish or Christian translators [Burnett,1996]. The degree of "disguise" within translations of results from Islam was likely commensurate to how important the disguised item was and the beeswax technology was important to North-Africa at that time.

In retrospect, when considering the advance Muslim scholars had over Europeans in Fibonacci's time, it must be realized that a significant part of Fibonacci's results are unavoidably efforts in translation. These translations were "filtered" by the church authorities and consequently the "translators"

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2. From the Islamic perspective at least, this would certainly be the case. However, this is the time of the Crusades and the Christian side would view this in a very different light. For instance, there is earlier claim that, in a manner similar to Sir Richard Burton, Adelard of Bath disguised himself as a Muslim student and stole a copy of *Euclid's Elements* before translating it from Arabic into Latin [RouseBall,1908,p50-62].

3. It is a historical fact which scholars these days do not like to mention : Kepler was a bona-fide astrologer as well as an astronomer.

had to refrain from a close association with Muslims or Muslim thought, which could pose a danger to themselves and their works. Often, results had to be "disguised". Nonetheless, with this understanding, Fibonacci's work provided an invaluable service in bringing significant mathematical contributions from the Muslim world to Christian Europe.

2.2.2. Mathematical Issues

As mentioned before, although Fibonacci provided his sequence in his *Liber abaci*, he does not provide any recurrence relation. Nonetheless Fibonacci provided a "model" by which to generate these numbers, which is equivalent to the recurrence relation itself, though Fibonacci provided no formulation for it.

We have argued in the previous section that Fibonacci was not aware of the Chinese triangle. In the unlikely event however that he was aware if it through the work of Omar AL-Khayyam, it is still highly unlikely that he could have inferred this "model" from the Chinese triangle itself. The diagonal sums of the triangle entries indeed produce the Fibonacci numbers but yields *no* "model". Generating the Fibonacci recurrence relation of (2) from the binomial coefficients of (7) is a straightforward task today, but in view of the previous discussion based on historical grounds, far too advanced given Fibonacci's knowledge on Mathematics.

Even though Fibonacci's rabbit reproduction model is artificial i.e. : it is *not* representative of the actual physical reproduction of rabbits, it does provide nonetheless a mathematical "model" for generating the Fibonacci sequence in complete agreement with the recurrence relation (2). Also, when one considers the various ways of generating this sequence that are known today, the reproductive model is the easiest one, involving mathematical manipulations that could be handled by Fibonacci.

In the next section, we provide an alternative conjecture that addresses the historical and mathematical issues associated with the "standard" conjecture.

3. "Alternative" Conjecture

3.1. "Exhibit A" : Mercantile Culture of Bejaia and the Bee "Family Tree"

A first step in establishing this conjecture is in identifying Fibonacci's environment during his period in North Africa. At that time, North Africa and

Spain were in a "golden age" under the Berber ruling dynasties of the Almoravids (11^{th} to 12^{th} centuries) and to a lesser extent, the Almohads (12^{th} to 13^{th} centuries). Bejaia had reached a peak as a major center in North Africa with a very significant intellectual élite, an artistic class and the equivalent of a wealthy bourgeoisie [Marcais,1986,1,1204-1206].

Amongst its chief commercial exports, beeswax figured prominently as Bejaia had one of the most efficient "technologies" for wax production during the middle ages [Marcais,1986,1,1204-1206, p.1204]. Indeed the French word for a certain type of candle called "bougie" derives from the word "Bejaia" (which is still called "Bougie" even today by many French people). This wax became very much in demand by members of the Christian clergy for their religious gatherings and ceremonies [4]. Produced by the Berber tribes known as the Kabyle within their mountains, this wax would be sold to Europe through the various merchants operating near the Mediterranean port of Bejaia [Brett&Fentress,1997, p.130]. Without a doubt, as part of the Pisan trade colony in Bejaia, Fibonacci was well aware of this technology and its commerce [5].

We note that although the rabbit reproduction problem is not realistic, Fibonacci numbers fit *perfectly* to the reproduction ancestry of bees. Within a colony of bees, only the queen produces eggs. If these eggs are fertilized then female worker bees are produced. Male bees, which are called drones, are produced from *unfertilized* eggs. Female bees therefore have two parents, drones in contrast, have just one parent. Looking at the family tree of a male drone bee (Figure 3a) we note the following :

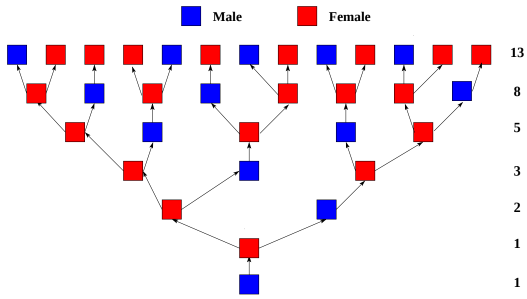
1. The male drone has one parent, a female.
2. He also has two grand-parents, since his mother had two parents, a male and a female.
3. He has three great-grand-parents : his grand-mother had two parents but his grand-father had only one, and so forth ...

4. Candles or torches based on animal fat were well known in Europe but these gave an unpleasant stench ; a highly undesirable feature during a religious ceremony.

5. Naturally, Europeans and in particular monks, would eventually improve their own beeswax "technology". During the middle ages, one of the most important jobs in an Abbey would become that of the "Beekeeper", as a huge quantity of wax was constantly required for the ceremonial candles. Bejaia would fall into disrepair and ruin after its conquest by the Spanish and subsequent domination by the Turks.

	generation	drone	worker/queen
A	1	1	2
n	2	2	3
c	3	3	5
e	4	5	8
s	5	8	13
t	6	13	21
o	7	21	34
r	8	34	55
s	9	55	89

(a) Bee Family Tree



(b) Fibonacci numbers .

Fig. 3. Bee "Family Tree".

By tracing the number of ancestors at each generation, one obtains *exactly* the Fibonacci sequence [Basin,1963,1,53-57] as shown in Figure 3b derived from the Table in Figure 3a. As we can see, the ancestry of a worker or even a queen [6] is simply a shifted Fibonacci sequence because of its connection to the ancestry of the bee drone. From a mathematical point of view, it is important to note that the number of ancestors at each generation n for (mammalian) sexual reproduction is simply 2^n. The ratio of two consecutive generations is asymptotically equal to 2 :

$$\lim \frac{2^{n+1}}{2^n} = 2 \; , \tag{9}$$

whereas in the case of bees, it is asympotically equal to the Golden number ϕ :

$$\lim \frac{F_{n+1}}{F_n} = \phi \approx 1.618 \; . \tag{10}$$

In other words, the ancestry trees for bees and rabbits do *not* have the same mathematical complexity. In tracing the ancestral family trees of rabbits or bees, the reader may note that we have traced the reproduction aspect going backwards in time rather than going forwards. Tracing the family tree is easy. To be able to model reproduction happening forwards in time in a realistic

6. Both the worker and the queen are females, the main difference is that the queen can reproduce because she is raised on "royal jelly".

fashion, one has to take into account sizes of litters as in the case of rabbits or yield of eggs in the case of bees and all these depend on statistical variations and conditions related to factors such as food, death toll and environment. In relative terms, this is a rather complicated problem. Naturally, one usually follows the easiest path to an answer.

Here we have a simple reproductive/ancestry model which perfectly fits the Fibonacci numbers and also falls into the mold of commerce-inspired problems which appear in *Liber abaci*; the rabbit problem - as Fibonacci presented it - being simply a variation (or disguised version) of the bee ancestry model. We wish to emphasize that the connection between the latter and Fibonacci's numbers is natural and perfect as opposed to the contrived artificiality of Fibonacci's rabbit problem.

Apart from the mathematical fit, it is essential to establish whether or not the Muslims in the time of Fibonacci could have sorted out the bee ancestry trees. To some, this might seem as a challenging statement. However, in the following sections we add to our present mathematical data, historical evidence (hitherto referred to as "exhibits" with the present section counting as "exhibit A") demonstrating that Fibonacci's numbers most certainly could have been inspired by the beeswax mercantile environment of Bejaia (Bougie).

3.2. "Exhibit B" : Translation Activity

Following the capture of Toledo during the "Reconquista", a large collection of books written in Arabic and Hebrew fell in the hands of the conquering Christians. These works were then translated by Christian scholars. As mentioned already, Fibonacci's work should be considered in the context of these translation activities, first centered around Toledo, a town inhabited at the time by a mixed population of Christian, Jews and Muslims living together side-by-side.

With the help of Jewish and Muslim scholars, the translation activities of Gerard of Cremona were continued by followers into the thirteenth century. This period marks the appearance of a major translator known as Michael Scotus (1175-1235 CE) [Thorndike,1965, Burnett,1994,2,101-126] (Latinized version of Michael Scot). Scotus became part of history and legend as the court "astrologer" of Frederick II, ruler of the Holy Roman empire. Scotus had learned greatly from the Muslims in areas of astrology and astronomy, alchemy, medicine and algebra. Although well viewed by the papal authorities around 1227, he would acquire the sinister reputation of a wizard and would be condemned in the inferno in Dante Alighieri's epic poem, *The Divine Comedy* (albeit "rescued" much later on in Sir Walter Scott's poem *Lay of the Last Minstrel*).

Scotus and Fibonacci were members of the court of Frederick II and would play their part in transmitting much of the scientific knowledge of the Muslims (largely from Moorish Spain) into Europe (largely Italy and Sicily), thereby planting many of the seeds of the Italian "Renaissance" [Haskins,1927, Burnett,1994,2,101-126]. Not only were Scotus and Fibonacci contemporaries, Fibonacci himself issued a revised version of his *Liber abaci* in 1227 CE [Burnett,1996] [7], with the following preface *dedicated to Scotus* [Thorndike,1965, IV, pp.34-35] :

> *You have written to me, my Lord Michael Scotus, supreme philosopher, that I should transcribe for you the book on numbers which I composed some time since. Wherefore, acceding to your demand and going over it carefully, I have revised it in your honor and for the use of many others. In this revision I have added some necessary matters and cut some superfluidities. In it I have given the complete doctrine of numbers according to the method of the Hindus, which method I have chosen as superior to others in this science ...*
>
> *To make the doctrine more apparent, I have divided the book into fifteen chapters, so that the reader may more readily find whatever he is looking for. Furthermore, if in this work inadequacy or defect is found, I submit that to your emendation.*

This preface is unusually flattering, almost the kind of acknowledgment a graduate student would give his doctoral supervisor and some have wondered about its true meaning or justification [Brown,1897]. There are other links between Fibonacci and Scotus : Scotus's use of the Pisan calendar [Burnett,1994,2,101-126, p.116-117] and the dedication itself suggests a connection between Scotus and Pisa [Haskins,1927, p.275,290].

7. Rashed [Rashed,1994,2, p.147] quotes 1228 CE as the year of dedication.

By a deductive process, we wish to discern the possible relationship between Fibonacci and Scotus. Judging from Fibonacci's other dedications and the practices of his time, such a dedication to two types of individuals :

1. A patron, someone affluent and usually part of the nobility. Frederick II was himself a patron. He was very impressed by Fibonacci and provided him with support. Dominicus Hispanus, a nobleman was another patron who had introduced Fibonacci to Frederick II himself around 1225 CE.
2. Someone who has been the source of inspiration to a mathematical question or challenge, as is the case of John of Palermo (who had posed a Diophantine-like problem which Fibonacci resolved in is *Liber Quadratorum*) or Theodorus of Antioch (mentioned earlier).

We know that Scotus certainly could not figure in the first category. He himself relied on benefices from the clergy [Thorndike,1965] and Frederick II himself to whom all of Scotus's works were dedicated [Thorndike,1965].

Firstly, we can be almost certain that Scotus knew enough mathematics to warrant Fibonacci's respect. As hinted by the dedication itself, Scotus would naturally have been interested in using the Hindu-Arabic numbers for his astrological computations in the service of Frederick II. However, we would like to draw attention to a different area. In Scotus's own book *Liber Introductorius*, which amongst other things, discusses matters that fall into the scope of Aristotle's *Meterologica*, Scotus presented an extension of Aristotle's five "regions" of air (dew, snow, rain, hoarfrost and hail) to include two other "regions", namely honey and laudanum (tincture of opium which was used as a pain-killer in Muslim medicine).

According to Scotus, "honey drops from the air on to flowers and herbs, and is collected by bees". However, as pointed out by Thorndike [Thorndike,1965, VI], Scotus distinguishes this "natural" variety from the artificial, produced, as he thinks, by the bee's *digestive process.* Scotus is amongst the first Europeans of the middle ages to make these observations and although by today's standards, neither statements are accurate (flower nectar rather than honey itself is indeed collected by bees in the "honey sacks" of their esophagi, and the digestive enzymes of their gastric saliva play a critical role in the "chemical" production of honey), it represented a step ahead in his time [8] as it must

8. Thorndike's analysis [Thorndike,1965, VI] shows that Thomas of Cantimpré made considerable use of Michael's work in his own writings on bees without giving much ac-

be reminded that many of the classic writings had been destroyed during the Barbarian invasions. Aristotle's zoological books on animals (*Historia animalium, De partibus animalium* and *De generatione animalium*) would not reappear in Europe until Scotus completed his own translations from the Arabic to Latin some time before 1220 [Thorndike,1965]. Scotus's sources are therefore primarily Muslim and indirectly Aristotelian.

3.3. "Exhibit C" : Background Knowledge concerning the Beehive and the Reproduction System of Bees

A very essential piece of our conjecture, perhaps the most essential, is establishing that the Muslim culture of Bejaia could have generated the bee family tree (Figure 3a). This requires the knowledge that

a bee drone results from an *unfertilized* egg.

Even with a lame numbering system (and we know the Muslims had better) : once this notion is recognized, it is very easy to tabulate the family trees of bee drones and obtain the Fibonacci sequence, as given in Table in 3a, to *any* order.

Naturally, the important question is : did the culture of Bejaia (Bougie) recognize *parthenogenesis*, i.e. *asexual* reproduction from an unfertilized egg ? This seems like a challenging proposition especially as the genetics of bee reproduction has only been worked out in the 20th century.

Although parthenogenesis (from a Greek word meaning "virgin birth") is claimed to be have been discovered in the 18^{th} century by Charles Bonnet(1720-1793), asexual reproduction was recognized as early as by Aristotle himself, who it must be noted, was an avid beekeeper. For that matter, apiculture can be traced back even earlier to ancient Egypt around 2400 BCE [Crane,1983]. Although it is true that the science of genetics is recent, the "art" of apiculture has been around since the dawn of civilization and it is worthwhile investigating just how developed it was by the time of Fibonacci.

knowledgment.

At this stage, we must open a large bracket as to what Aristotle himself knew and wrote about bees in his book *Historia animalium* (History of animals). His knowledge was considerable [Aristotle,1995,1-11] and some of his hypotheses and conclusions were fairly accurate for his time. Long before the invention of the microscope, Aristotle could correctly distinguish the 3-member caste system of the bees : workers, drones and one ruler. He correctly described many aspects of the development of the bees in the immature stages. He also wrote that bees had "a keen olfactory sense" [Aristotle,1995,1-11, p.705, cit. 444^b 7-12], as vindicated by the fact that bees use odor (chemical trails) as a communication tool. However,

1. Aristotle misunderstood the gender of the ruler. He believed the ruler was a male (king) and not a female (queen) [9].
2. Aristotle knew that bees obtained material from flowers but he suggested that honey was actually deposited from the atmosphere (a belief incorporated into the writings of Scotus, as seen in the previous section).
3. Aristotle misunderstood the reproduction system of bees.

Experienced beekeepers know very well that if a queen becomes old or afflicted with disease, she can no longer mate with drones. Furthermore, if a queen dies, some workers become "pseudo queens" and lay eggs. However, as these "pseudo queens" are unable to mate (only true queens can), the resulting eggs are also unfertilized. In either case, there is an increase in drones at the expense of workers and the hive is in serious danger of self-destruction. A balanced population made of a majority of workers and a sufficient minority of drones is needed to maintain the dynamic equilibrium of the bee hive.

Aristotle was actually able to observe the resulting brood of drones appearing in these circumstances, and it is important to note that such an observation was possible. However, he failed to draw the right conclusions. Instead, he believed that bees do not give birth but fetched their young from flowers (spontaneous generation). However, it is vital to note that Aristotle *also* wrote [Aristotle,1995,1-11, p.872, 553^a $32 - 553^b$ 1] :

9. Aristotle's erroneous notion that workers bees were male as stated in (1) was a common belief in Europe by the time of Shakespeare as can be testified by his play, "Henry the Fourth", where some of the characters speak about bees as soldiers led by a king [Shakespeare,1914, part2,Act IV, Scene 5].

> "Others again assert that these insects (bees) copulate, and that drones are male and bees female."

indicating that the alternative notion of a female bee ruler existed at the time of Aristotle, as can be testified by Greek mythology [10]. Moreover, the idea of parthenogenesis is mentioned in a number of instances within Greek mythology as in, for example, a particular version of the birth of the god Hephaistos from Hera [Graves,1990, I :12.c] and the birth of the creature Ladon from Mother Earth [Graves,1990, II :133.b].

In spite of Aristotle's misinterpretations, one could see that even in his time, reliable observations on bees were possible. In hindsight, we can see that had it not been for his belief that the bee ruler had to be male, Aristotle's observations and knowledge of sexual and asexual reproduction could have potentially lead him to the realisation that (male) bee drones resulted from unfertilized eggs. All the needed "ingredients" for this realisation were present within his writings. To reiterate :

> Once the genders of the bee 3-member caste system are properly sorted out and noting that :
>
> 1. Aristotle's accurate observation of bee (drones) hatching without fertilization
> 2. Aristotle's knowledge of asexual reproduction,
>
> one is inexorably guided to the realisation that the male bee drones simply resulted from unfertilized eggs.

Given that Aristotle faced opposition to his beliefs by his contemporaries, it becomes tantalizing to consider that someone could have made the realisation long before the middle ages. We will return to this point later.

As a side issue, we also note that the occurrence of twins (used in Fibonacci's rabbit model) and parthenogenesis (appearing in bee reproduction) are both natural forms of "cloning". The mathematical connection between Fibonacci's rabbit model and the bee ancestry tree can therefore also been seen

10. Melissa was identified as the Queen Bee who annually killed her male consort (much as the bee drone dies at copulation) [Graves,1990, I :7.3]. Her priestesses were called Melissae. See also [Graves,1990, I :18.3] concerning Aphrodite Urania and the tearing out of sexual organs of the male which is indeed descriptive of what happens to a bee drone at the time of mating.

in this light [11]. Aristotle himself devoted considerable thought to the aspect of twins in his "History of animals" Michael Scotus also appears to have been fascinated by twins [12].

From the third until the eleventh century, biology was essentially a Muslim science, as the Roman empire crumbled under the onslaught of barbarian invasions. Muslims had discovered the works of Aristotle and Galen, translated them into Arabic, studied and wrote commentaries about them. *Al-Jahiz* (776-868 CE) (e.g. see [Zahoor,2000]), is a particularly noteworthy Arab biologist. In his *Kitab al-hayawan* ("Book of Animals'"), in which he reveals some influence from Aristotle, the author emphasizes the unity of nature and recognizes relationships between different groups of organisms.

It is worth noting that all the pertinent apiculture notions of the Islamic middle ages appear right in the Koran [Toufy,1968]. In a section of the Koran "Surah an-Nahl" $(16 : 68 - 69)$, which means "The Bee", it is stated that

> And your Lord inspired the bee, (Saying), "Take for yourself dwellings in hills, on trees and in what they (mankind) build. Then eat of all fruits." From their bellies comes a drink of varied colors, beneficial to men. This is a meaningful sign for thinkers.

In this extract, one recovers the origin of Scotus's mention of the role of the bee's digestive process in the making of honey. Obviously, Islamic beekeepers of Fibonacci's time believing in this notion naturally concluded that the yield of wax and honey depended as much (or more) on the number of bees, than the actual amount of flowers or plant resources. This would justify a mercantile impetus to understand the reproduction of bees.

What is vitally important is the gender of the bees as written in the original Arabic of this passage. In both verses, it uses *female* verbs in describing the bee, in Arabic : "fa'sluki" and "kuli" (for the imperative "eat"). Also the imperative "take" in this passage is the translation of the Arabic word "attakhidhi" and of feminine form (Arabic verbs, unlike English ones, differentiate between the sexes). Like French, the Arabic female form is used

11. Ironically, it is claimed that around the 1940's, experimental biologists succeeded in artificially stimulating rabbits to reproduce without fertilization. The rabbit problem can then be "rescued" but of course, Fibonacci could not have been aware of this !

12. As can be seen, for example, his gynecological case study of "Mary of Bologna" [Jacquart,1994, p.32].

when all those it refers to are female, whereas the masculine is used when a group contains at least one male. Thus, in this passage, *all* the bee workers are *female* and two of Aristotle's major misinterpretations are addressed :

1. The Muslims realized the bee workers were female and the drones were male.
2. The Muslims had a more accurate understanding of the actual production of honey by the bees by linking it to the bee's digestive process (a fact also mentioned by Scotus).

The above is confirmed by the apiculture writings of the Islamic scholars Al-Jahiz, and later by Al-Qazwînî (died 1283 CE), Al-Damîrî (died 1405 CE) and Al-Maqrîzî (died 1442 CE) [Toufy,1968]. This is already an improvement on Aristotle's writings on bees but the remaining question is : what is the gender of the Queen ?

Abû Dhu'ayb, a Hudhayli [13] poet and contemporary of the Prophet Mohammed, wrote about "the power of the *queen* in the bee city" [Toufy,1968, p.81] and one could think that the matter would be finally resolved. However, in mitigating both the writings of Aristotle and the tenets of the Koran, while the bee workers were definitely female, Al-Jahiz as well as most other Islamic scholars would speak of a bee "king" even though authors appearing after Al-Jahiz admitted the existence of a bee "queen" (and Al-Jahiz admitted the existence of "mothers"). This "king" was called the *ya'sub* or "stallion of the bees and the prince of the (female) bee makers" [Toufy,1968, p.62]. However, the romantic picture of the ya'sub given by Al-Jahiz would change dramatically. By 1371 CE, Al-Damîrî would declare [Toufy,1968, p.68] that when the honey supply became insufficient, the bee workers would *eliminate* the "king" and the males. This is fairly accurate : in winter or when honey is lacking, (female) worker bees eliminate (male) bee drones from the hive. Furthermore, the historian Al-Maqrîzî collecting the common knowledge of apiculture known in his time through the work of predecessors whose names he would not given, finally declared :

> "Some claim that the males build their own cells but the males do nothing. The work is done by the queens ; it is they who guide (i.e. dominate) their kings and their males."

13. The Hudhayli were a tribe in the Arabian Peninsula.

This is also fairly accurate and given the patriarchal nature of Islam at that time, quite an admission. The bee "king" still existed [14] but somewhere between the 9th century of Al-Jahiz and the end of the 14th century, there was a complete transfer of power from the king to the queen. Moreover, already by the time of Al-Qazwînî, the description of bee morphology was remarkably detailed including a full array of colors, shapes and other characteristics.

The reader may be understandably confused by the apparent contradictions (and double-think) within these Islamic writings of bees. However, these can be understood as follows. Most if not all cultures initially believed in a queen bee rather than a king bee - this was merely natural : they clearly identified the largest bee, say the "ruler", whose size was much larger than that of any other bee in the hive and with no apparent equal in stature of size or importance within the hive, as simply the "mother" of all bees. As maternity was perennially obvious and paternity perennially harder to establish or fully understand, this was simply natural and universal. Aristotle would be the first to write down mechanisms of sexual reproduction (as far as we know). The prevalence of a "queen bee" can be confirmed by citations in various cultures. To mention a few :

1. Greek mythology (prior to Aristotle) believed in the supremacy of a "queen bee".
2. A passage in the ancient Vedic writings of India called Prashnopanishada [Upanisads,1884/1963,2/15], dated at around 500 BCE, also mentions a "queen bee".
3. The warrior-woman Deborah mentioned in the Old-Testament (or Jewish Tanach) was a ruler whose name meant "queen bee".

Furthermore, the poetry of Abû Dhu'ayb confirms that people in the Arabian peninsula also believed in a queen bee up until the rise of Islam (and quite likely into the 8th century). Initially, everyone believed in a "queen" or "mother" bee but the inheritance of Aristotle's notions (quite possibly coupled to the patriarchal views of Arabic culture) prompted the Muslim scholars to believe in a bee "king" instead. However, the growing input from real life apiculture forced severe revisions on these notions. With a relax-

14. Many bee-keeping peasants in the Islamic world having preserved such notions from generation to generation still have legends about a bee king.

ation from Muslim orthodox principle, these revisions would be eventually admitted publicly as much as they were allowed to.

To summarize : from Aristotle to Al-Jahiz, one passes from a male majority populated to a female majority populated beehive and within a much shorter time interval, from Al-Jahiz to his successors, one passes from a male dominated to a female dominated beehive. How and why did this happen? The answer to this question cannot lie solely within these writings from the Arabian peninsula. Wax production (a safe indication of increased honey output) in the Arabian peninsula is not mentioned until the 15th century by Al-Maqrîzî while the wax technology of Bejaia was well established in Fibonacci's time (between the 11th and 13th centuries). This provides a clear indication that to answer this question one must look outside the Arabian peninsula.

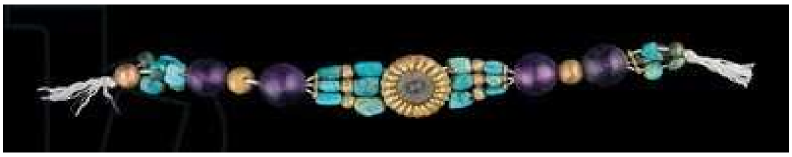
By the end of the 7th century, the spread of Islam in the Maghreb of North Africa had almost been halted by a Jewish-Berber queen known as Kahena [Beauguitte,1959] who lead a military coalition made of Numids, Moors, Jews and Christians including Romans and Egyptian Copts. Viewed as a second "Deborah", she became the model and symbol of the fiercely independent Amazigh women [15]. Although queen Kahena met defeat, remnants of this culture (and its defiance) would still exist by the 12th century, and resistance to Islamic protocols would continue as in, for example, cited violations of the Islamic code for woman's dress [Libas,1986,742-246]. None of the Almohad or Almoravid Berber rulers recognized the authority of the Caliphs of Baghdad. Some of the consequences of those "cultural" defiances and distinctiveness remain to this day [16]. In many respects, the Berber culture of North Africa was more advanced than that of the Arabian peninsula. Moreover, from the 10^{th} century or so and onwards, the Arabs were losing their grip and Islam became fragmented. By the time of Leonardo de Pisa, communications between the various realms of Islam were greatly reduced.

During the time of Fibonacci, the Maghreb acted as a "corridor" between the cultures of Spanish Andalusia and those east of Egypt. Bee apiculture had

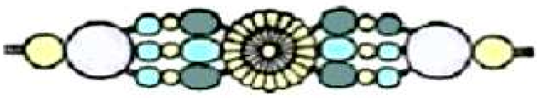
15. North African Berber tribe with a strong matriarchal element. Some associate the women of these tribes to the myth of the Amazons.

16. For example, amongst the North African Berber tribes known as the Tuareg, it is the men rather than the women who are veiled. .

Photograph



Jewelry dated de Djer, 1st Dynasty, Abydos



Crown Daisy with 21 petals

Fig. 4. Egyptian Djer Bracelet : modeled on flower rays. Courtesy in part of Graham Oaten. Many drawings of this bracelet exist and in some 22, rather than 21 petals (rays) are present.

started in Egypt and had spread westwards to North Africa and to Greece via Crete [Graves,1990, I :82.6] and was far more developed than in Rome or even Greece [Crane,1983].

3.3.1. Archaic Egypt

Since Amazigh bee apiculture has its roots in Ancient Egypt, the following question naturally arises : did the Egyptians know about the Fibonacci numbers ? To this end, we mention a thread pointed out by Graham Oaten : the artists of archaic Egypt observed and copied nature's patterns to produce some astonishing artifacts. In particular, we cite a gold bracelet [Kantor,1945], found in the tomb of a king at Abydos, presumably belonging to a queen of Zer (or Djer) and dating back to the first dynasty (around 3000 BCE). The bracelet is currently in the Egyptian museum of Cairo. This bracelet is made of a gold rosette centerpiece which resembles a modern watch like design. The floral pattern of this rosette (presumably a daisy) has exactly 21 rays ! as shown in Figure 4 (21 being a Fibonacci number) [17]. Whether or not the ancient Egyptians knew about the Fibonacci numbers or the Golden number is an endlessly controversial subject. Some like Axel Hausmann claims the ancient Egyptians knew both the Fibonacci and Lucas numbers (same recursion formula but different starting point) and he may well be right in arguing that the Fibonacci numbers are embedded in the original structure of the Aachen city hall ("die Rathaus") built around the 9th century [Hausmann,1995] as a residence for Charlemagne. However, most scholars still refuse to believe the ancient Egyptians knew these numbers though recently C. Rossi hesitates to draw any conclusions one way or the other [Rossi,2004]. We do not claim that the ancient Egyptians figured out the Fibonacci numbers, but given their universal presence in nature, it is quite possible that they recorded natural phenomena exhibiting these numbers.

The link to Bejaia can be appreciated thanks to the research of Helene Hagan [Hagan,2000] in the area of Amazigh history, folklore and etymology. Indeed the notion of a queen bee goddess is prominent in the folklore of

17. We must point out that the handmade reproduction of this floral pattern in many a reference is incorrect as it shows 22 and not 21 rays. However, a detailed examination of a picture of the original bracelet (e.g. see [Smith,1981,p45m]) shows that the number of rays is indeed 21, demonstrating how carefully faithful to nature were the artists of Archaic Egypt.

Fig. 5. Barberini Exultet Roll : The Praise of The Bees. Biblioteca Vaticana (Vatican City), Cod. Barb. Lat. 592.

Amazigh Berbers and can be traced to the archaic period of the pre-dynastic age of Egypt. This is not surprising as Egypt was originally the source of their beekeeping culture. This bee goddess is still worshipped within festivals in mountains of North Africa. There is every indication that the Kabyle Amazigh apparently understood the 3-member bee cast. However the actual details and ramifications tied to this research (some of which is still in progress) are too considerable to be addressed in detail here. This would require at least an article in itself. However, we can mention a few identifications. For instance, the word "tammnt" meaning honey existed in the archaic Egyptian vocabulary and the very same word is used today throughout Berber territory. This word in itself is very revealing because of its persistence across millennia and thousands of miles. It is feminine, both in the ancient Egyptian and the Amazigh languages.

3.3.2. Medieval Catholicism - Exultet Rolls

The notion of the "virgin birth", which may be related to the observation that male bee drones result from unfertilized eggs can also be found in the Koran, in reference to the birth of Jesus. The acceptance of parthenogenesis helped Muslims rationalize the birth of Jesus as something unusual but possible [18] without the "divine Father" Christians associated with Jesus himself [19]. Marcus Toledanus (Mark of Toledo), a colleague [Burnett,1994,2,101-126, p.104] of Michael Scotus in Toledo Spain, made a translation of the Koran from the arabic into Latin, and completed it around 1210 CE [Burnett,1994,2,101-126].

The subject of "virgin birth" the Exsultet or Exultet or Easter Proclamation, a hymn of praise sung before the Paschal (or Easter candle) during the ritual known as the Easter Vigil. Made out around the 12th century, the Exultet Roll of Salerno includes a section called "The Praise of the Bees" describing fascinating images of beekeeping in the Middle Ages. The text extols not only the marvelous skill of the bees who produce honey and wax from flowers, but also their reputed chastity leading Catholic belief of the Virgin Birth of Christ. Questions of the origin of the Exultet rolls are not fully resolved [Kelly,1996, p.206]. One possibility believed by some is a tradition descending from Augustine, himself a native of North-Africa. Others believe instead in a Byzantine influence dating no earlier than the 10th century. The Barberini Exlutet roll created in the Benedictine abbey of Monte Cassino (Italy) and dated at around 1087 CE also features the praise of the bees as shown in Figure 5.

3.4. Assembly of Evidence

At this stage, the conservative reader, in particular the mathematically minded one may feel somewhat bombarded by this strange mixture of mathematics, "magic" and religious beliefs. However, as explained before, this is a feature of Fibonacci's times [20] and we must try to follow the pattern of

18. With a bit of help from Allah, of course. Do note that contrary to Christian belief, Muslims believed Jesus was fatherless.

19. Without a doubt, skeptics will claim this is rationalization or reinterpretation "after the fact" but that is not the point : the bottom-line is an issue on how soon that "rationalization" is actually done.

20. If one insists, a modern view would argue that much of the thinking in the middle ages was undisciplined. As mentioned before : scientific fact coexisted with misinformation, superstition and religious beliefs.

thought (however questionable or faulty it may seem to be) that lead to their results. To reiterate, so far we have the following known facts :

– Fibonacci dedicates his *Liber abaci* to Michael Scotus even though the Emperor Frederick II was his patron (suggesting a debt of acknowledgement of some kind towards Scotus).
– The beeswax "technology" of Bejaia was well developed at that time and Bejaia was a major exporter of wax via the Pisan Trade colony where Fibonacci worked.
– Our analysis (Exhibit C) suggests that the intellectual and mercantile culture in Bejaia had just reached the level of sophistication to be able to work out and tabulate the ancestries of bees during the middle ages.
– The bee ancestry model *perfectly* fits the Fibonacci sequence but the latter is *not* representative of the true physical picture of rabbit reproduction by which Fibonacci originally presented his sequence in *Liber abaci*.
– It has been established that many of the algebraic problems in Fibonacci's *Liber abaci* are (disguised) translations of Mathematical results of Muslim scholars into Latin. These translations were made in Europe.
– Scotus wrote about bee apiculture and his sources are Muslim and Aristotelian.
– A major "school" of translation in Fibonacci's time was in Toledo in Castille, Spain where Scotus was based before 1220. In particular, Scotus translated Aristotle's works in zoology.
– Essential notions of Muslim beekeepers during the middle ages are expressed in the Koran which was translated into Latin by Marcus Toledanus, a colleague of Scotus.
– Beeswax was in great demand for candles by the clergy. Michael Scotus and Marcus Toledanus worked for the church as both were associated with the Cathedral at Toledo [Burnett,1994,2,101-126].

We submit that all this may be more than an unusual coincidence and these various facts might instead assemble into a puzzle whose image becomes clear :

> It is suggested here that the Fibonacci number sequence originated within the framework of a reproductive model from the intellectual and mercantile culture of Bejaia in North Africa. It is also suggested that this involved a collaborative effort between himself and Michael Scotus (within the Toledan school of translation) in the light of translations from Muslim scholarship into

Latin as well as exchange of information.

Fibonacci naturally focused on mathematical aspects and Scotus was interested in the biological ones. In itself, this could well provide a plausible origin of the Fibonacci sequence and some of the reasons behind Fibonacci's dedication to Scotus in his *Liber abaci*.

3.5. "Exhibit D" - Fibonacci Numbers and Ancient Greece

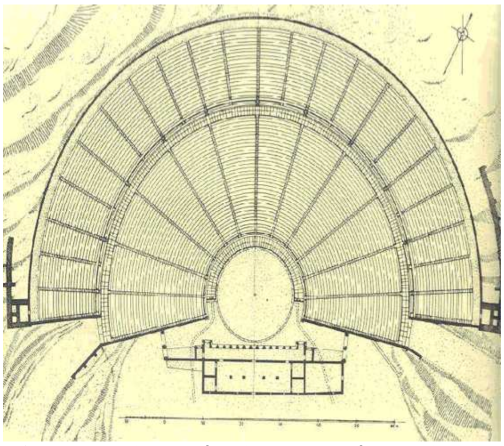


Fig. 6. Plan of Theatre of Epidaurus in Argolis.

An important implication in our conjecture is that if the Greeks had overcome Aristotle's misinterpretations and recognized parthenogenesis, the Fibonacci numbers could have potentially been derived within the framework of a bee reproduction model as far back as in the time of Aristotle. At first, this would seem extraordinary as we have, so far, found no direct written records of such a discovery. We know the Fibonacci sequence appears at a much earlier date, in Indian mathematics, in connection with Sanskrit prosody [Singh,1985,12-3,229-44, Knuth,2006,4,100, Knuth,1968,1,100] but in the context of music. There is however architectural evidence in Greece (dating from the Hellenistic period) indicating that the ancient Greeks did know the Fibonacci numbers after all. The evidence can be seen by visiting the site of the theater of Epidaurus in Argolis [Charitonidou,1978, p.38-47],[Iakovidis,2001, p.130-133] (dating from the Hellinistic period) as illustrated in Figures 6, 7a and 7. Figure 6 shows the theater plan. The most notable occurrence of Fibonacci numbers is that the theater consists of 34 rows and then an additional 21 (Figures 7a and 7b) built around the 2nd century BCE. Both 21 and 34 are Fibonacci numbers. This aspect is emphasised by the Greek author (Dimitris Tsimpourakes) [Tsimpour'akhs,1985, p.231]. Interested in mathematics and architecture, he reasoned that the ancient Greeks tried to inject "harmony" into Greek architecture much along the lines of what was already done with the Parthenon, the ratio between the Fibonacci numbers 34 and 21 providing an approximation of the Golden

(a) Lower level : 34 rows of seats.

(b) Upper level : 21 rows of seats.

Fig. 7. Theatre of Epidaurus. Picture taken by T.C. Scott, December 2003.

number :

$$\frac{34}{21} \approx 1.619 \approx \phi \tag{11}$$

In a meticulous analysis by Arnim von Gerkan and Wolfgang Muller-Wiener [Gerkan&Muller-Wiener,1961], extrapolation of the lines defining the aisles joining the rows of seats of the theatre to its center reveals two back-to-back *Golden triangles*, namely triangles balanced by the Golden number. These are in the shape $\triangleleft$ and $\triangleright$ together forming a diamond shape located just below the center of a pentagon, as shown in Figure 8. This construction by Gerkan takes into account slight irregularities and asymmetries likely caused by earth tremors and ground movements over the last 2500 years. Each Golden triangle is an isosceles triangle where the apex angle is :

$$\theta = \cos^{-1}\left(\frac{\phi}{2}\right) = \frac{\pi}{5} = 36^\circ \tag{12}$$

and the bases angles are therefore each 72°. One may well consider this as a message left by these ancient architects!

Tsimpourakas also cites, although less convincingly, the numbers 19, 15 and 21 embedded within a theater at Dodona in Epirus in northern Greece (9a

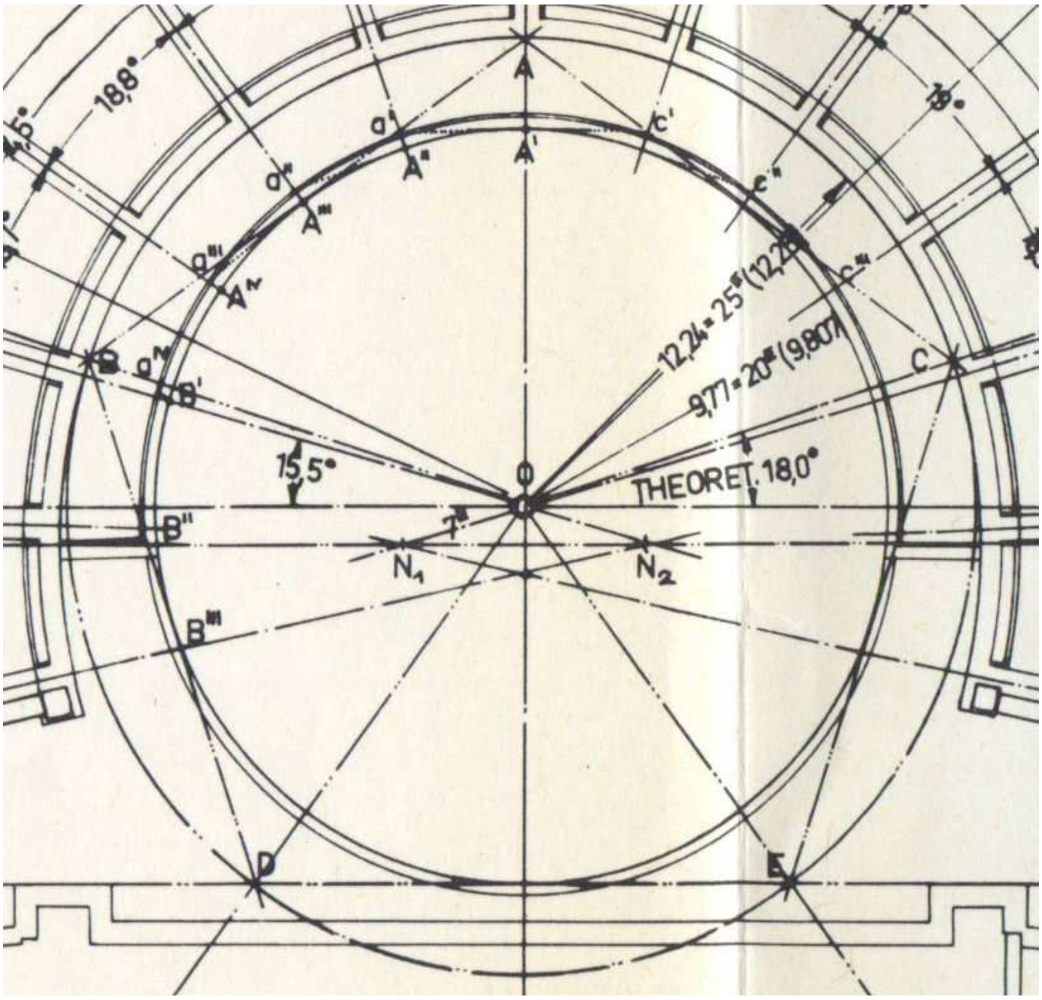
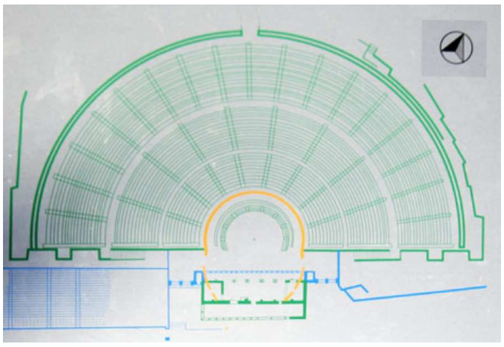


Fig. 8. Golden Triangles near Center of Theatre of Epidaurus.Extract from [Gerkan&Muller-Wiener,1961].



(a) Plan. **(b)** Picture.

Fig. 9. Theatre of Dodona.

and 9b), which may used to approximate the Golden number :

$$\frac{19+34+21}{19+15} = \frac{19+15}{21} = \frac{34}{21} \approx \phi \ . \tag{13}$$

In this calculation, the Fibonacci numbers 21 and indirectly $34 = 19 + 15$ appear. However, the case can be made a bit more convincing when examining the plan of the theatre of Dodona. We see that ten radial staircases divided the koilon, involving the first two sets of 19 and 15 rows, into 9 *cunei* (tiers or wedges of seats). The upper part of 21 rows has intermediate staircases and 18 *cunei* or tiers. It is also separated by the lower set of 34 rows by a wider gangway. Thus the design of the theatre suggests that the first two sets of 19 and 15 rows form a near continuous set. Though this might only be coincidence, the ratio 19/15 is a very good approximation of the square root of the Golden number, $\sqrt{\phi} \approx 1.27\ldots$. This ratio is reminiscent of the Egyptian triangle claimed to be embedded in the proportions of the Great pyramid [Tsimpour'akhs,1985] which is not surprising given the mythological connection between Dodona and Egypt cited by the historian Herodotus. The Dodona oracle was established by two priestesses from Thebes in Egypt, who were abducted by Phoenicians, and turned into two black doves. These were *Peleiades* who founded the sanctuary of Zeus Ammon in Libya located in the oasis of Siwa and cited by Amazigh/Kabyle legends [Hagan,2000] and the Oak-tree cult at Dodona.

As mentioned before, as testified by the work of Euclid and Pythagoras, the Greeks knew about the Golden ratio ϕ (3) which can be expressed as the root of :

$$\phi^2 = \phi + 1 \tag{14}$$

Next, if we multiply equation (14) by ϕ itself, we get :

$$\phi^3 = \phi^2 + \phi = (\phi + 1) + \phi = 2\ \phi + 1 \tag{15}$$

where ϕ^2 was replaced by the right side of (14). If we then multiply (14) by ϕ^2 and use (15),

$$\phi^4 = \phi^3 + \phi^2 = (2\ \phi + 1) + (\phi + 1) = 3\ \phi + 2 \tag{16}$$

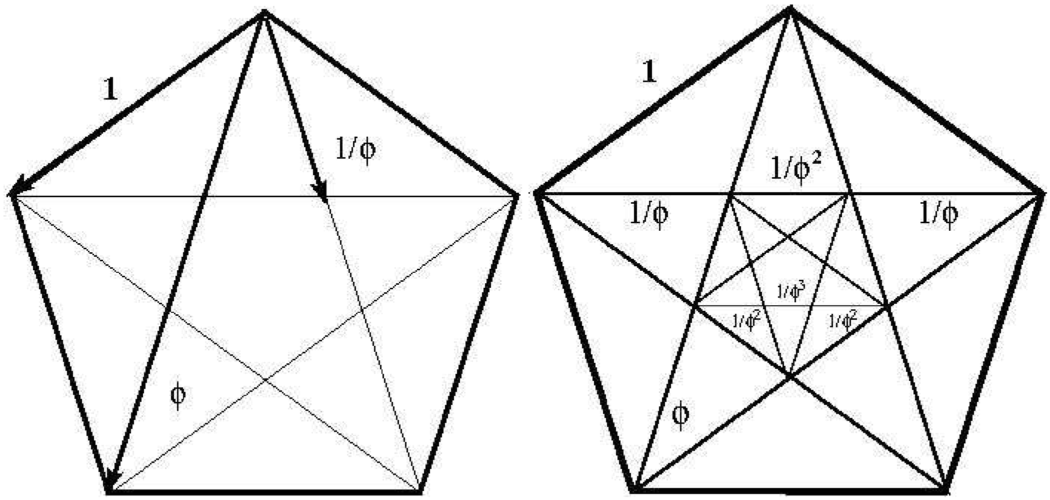
Similarly,

$$\begin{aligned}
\phi^5 &= \phi^4 + \phi^3 = (3\ \phi + 2) + (2\ \phi + 1) = 5\ \phi + 3 \\
\phi^6 &= \phi^5 + \phi^4 = (5\ \phi + 3) + (3\ \phi + 2) = 8\ \phi + 5 \\
\phi^7 &= \phi^6 + \phi^5 = (8\ \phi + 5) + (5\ \phi + 3) = 13\ \phi + 8 \\
\phi^8 &= \phi^7 + \phi^6 = (13\ \phi + 8) + (8\ \phi + 5) = 21\ \phi + 13 \qquad (17) \\
\vdots & \quad \vdots
\end{aligned}$$

One can notice from the right hand side of these equations that ϕ^n can be written *linearly* in terms of ϕ and the Fibonacci numbers, as well as the process of recursion itself. Admittedly the derivation is *algebraic* (something the Muslims could have worked out) rather than geometric (and the Greeks would have followed geometric arguments). The question is : how could the Greeks have generated the Fibonacci numbers by geometrical means ? In the following, we outline a geometrical derivation that answers this question.

By examining the Golden square [Bicknell&Hoggart,1969,7,73-91], one can geometrically build up a relation as high as ϕ^4. Since the discovery of irrational numbers by Hippasos [21], a member of the Pythagorean school of mathematics (5^{th} Century BCE), within the incommensurability or irrationality of the diagonal in the pentagon or the pentagram (the very symbol of the Pythagorean school itself), the Golden ratio ϕ plays an essential role. One can see the incommensurability (irrationality) by looking at a pentagon and the one formed by all its diagonals. As shown in Figure 10a, the ratio between

21. Legend has it that the disciples of Pythagoras were at sea and Hippasos was thrown overboard for having the "heresy" of producing an element in the universe which denied the Pythagorean doctrine that all phenomena in the universe can be reduced to whole numbers or their ratios [Kline,1972/1990].

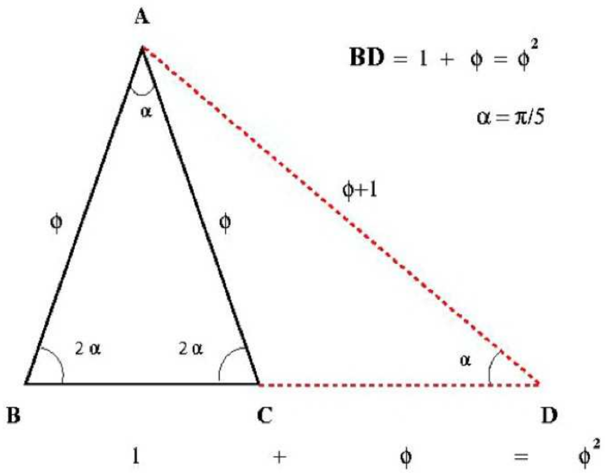


(a) Pentagon and Inner Pentagram. **(b)** Recursive creation of pentagons.

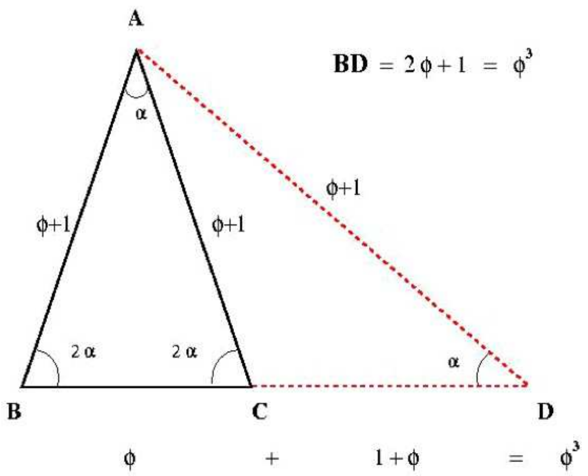
Fig. 10. Pythagoras : Recursive use of pentagons naturally balanced by Golden Number.

the diagonal of a pentagon and its side is equal to ϕ. By inserting more and more diagonals into Figure 10a, we get the divisions of the pentagon into smaller sections as shown in Figure 10b. Note that each larger (or smaller) section is related by the ϕ ratio, so that a power series of the Golden ratio raised to successively higher (or lower) powers is automatically generated : ϕ, ϕ^2, ϕ^3, ϕ^4, ϕ^5, etc . . .In this manner, the derivations in (17) can find their geometrical equivalent.

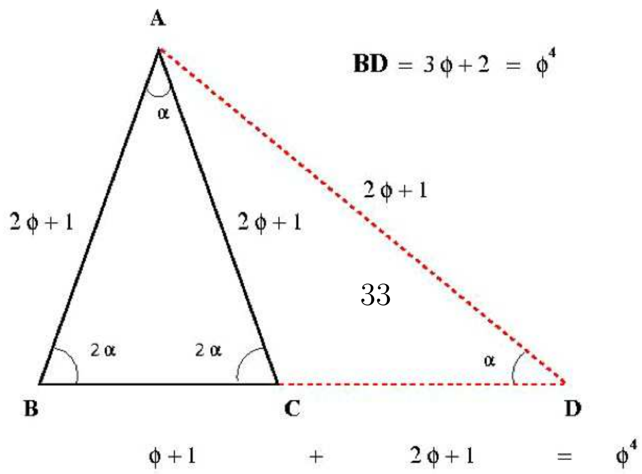
The easiest way to demonstrate this point is to begin by considering the construction in Figure 11a. In this Figure, the triangle ABC is isosceles ; in other words, the distances AB and AC are equal as are the two angles ABC and ACB. The lengths AB and AC are equal to the Golden number ϕ and the length BC is unity. The angles ABC and ACB are each equal to twice the angle BAC. The sum of the three angles inside a triangle is equal to 180 degrees or π radians ; consequently, the angle BAC is equal to 36 degrees or $\frac{\pi}{5}$ radians. The latter is expressed in modern terms but, nonetheless, as mentioned by Heath [Heath,1931], the Pythagoreans knew that the sum of the angles inside a triangle is equal to the sum of two right angles.



(a) First Triangle



(b) Second Triangle



(c) Third Triangle

We then extend the side BC until a point D, such that $CD = \phi$. Since $AC = CD = \phi$, the triangle ACD is also isosceles by construction and the angles CDA and DAC are as a result equal. Since $CD = AB$, we have :

$$\frac{CD}{BC} = \frac{AB}{BC} \tag{18}$$

The angles ACB and DCA are complementary and therefore their sum is equal to 180 degrees or π radians. The angle DAC is equal to 36 degrees by construction. Thus the line AC bisects the angle BAD. Furthermore, since the angles DAB and ABC are the same (and equal to 72 degrees for the modern reader), the triangles ABC and DAB are *similar* since their respective angles are equal. That is, the original isosceles $36-72-72$ degree triangle ABC is embedded in a second isosceles $36-72-72$ triangle DAB and is similar to it. Since the sides of two similar triangles that lie opposite to equal angles are proportional,

$$\frac{AB}{BC} = \frac{BD}{AB} \tag{19}$$

By combining (18) and (19) and using $AB = CD$, we obtain

$$\phi = \frac{CD}{BC} = \frac{AB}{BC} = \frac{BD}{AB} = \frac{BD}{CD} \tag{20}$$

If we let $BC = 1$, then $CD = \phi$, since $CD/BC = \phi$ and $BD = BC + CD = 1 + \phi$. Substitution of the values for BD and CD in 20 yields :

$$\frac{\phi}{1} = \frac{1+\phi}{\phi} \quad \text{i.e.} \quad \phi^2 = \phi + 1 \tag{21}$$

This is the equation whose positive root defines the Golden ratio.

We now repeat this exercise, this time by taking the outer isosceles triangle DAB and making it play the role of the first isosceles triangle ABC (Figure 11b). Within the "new" isosceles triangle ABC, $AB = AC = \phi^2 = \phi + 1$ and $BC = \phi$. In this triangle we now extend the side BC, which is opposite to the angle that is equal to α, by a length CD equal to $\phi^2 = \phi + 1$. Since the angles of Figure 11b are equal to those in Figure 11a, a similar analysis leads to the same proportions as expressed in equations (16) and (17). In particular, generalizing from equation (20),

$$BD = BC + CD = \phi * CD \tag{22}$$

By construction of the outer isosceles triangle, the total length BD is given by

$$BD = BC + CD = \phi + (\phi + 1) = 2\phi + 1 \tag{23}$$

However, from equation (22), the length BD satisfies

$$BD = \phi\, CD = \phi * (\phi^2) = \phi^3 \tag{24}$$

and consequently

$$BD = \phi^3 = 2\phi + 1 \tag{25}$$

which is indeed equation (15).

We now iterate once again (Figure 11c). In the "new" isosceles triangle ABC, $AB = AC = \phi^3 = 2\phi + 1$ and $BC = \phi^2 = \phi + 1$. We repeat our exercise and draw a line from point C to point D, this time of distance $CD = \phi^3 = 2\phi + 1$. In this case, use of equation (22) yields

$$BD = \phi\, CD = \phi * (\phi^3) = \phi^4 \tag{26}$$

Further, from the construction of the outer isosceles triangle

$$BD = BC + CD = (\phi + 1) + (2\phi + 1) = 3\phi + 2 \tag{27}$$

and consequently

$$BD = \phi^4 = 3\phi + 2 \tag{28}$$

which is indeed equation (16). The above identity has been obtained as a result of two iterations. Repeating this exercise once more yields

$$BD = \phi^5 = 5\phi + 3 \tag{29}$$

A further repetition of the geometrical construction yields

$$BD = \phi^6 = 8\phi + 5 \tag{30}$$

as in (17). The Fibonacci numbers 21 and 34 are now within the reach of three more iterations. Thus, starting with an isosceles triangle embedding the Golden ratio, a *recursive* embedding of isosceles triangles into larger and similar isosceles triangles reproduces the results obtained from the algebraic derivation of the previous subsection. This approach is known as the *Gnomon* and it is believed that Pythagoreans knew how to apply this approach to isosceles triangles [Thompson,1992, p.761-763]. It can be shown that the recursive embedding of these isosceles triangles yields a logarithmic spiral, but this falls outside the scope of the current investigation.

Note that the recursion adopted so far is based on the construction of larger triangles. A recursion in a backwards direction, i.e. taking the outer triangle of Figure 11a and bisecting the angle DAB to create the inner isosceles triangle CAB, could have also been followed. An examination in particular of the pentagon and the smaller sections caused by the divisions of Figure 11b shows that the major “building” blocks of these sections are indeed triangles. The type of analysis that we have followed here could therefore have been applied to these progressively smaller sections. Note that the pentagram was the very *symbol* of the school of Pythagoras which, in addition to being a mathematical school of thought, was also a mystical and secret society. Not surprisingly, the use of this symbol throughout the centuries has often been associated with mysticism and witchcraft.

Our geometric derivation makes use of knowledge contained within Book IV of Euclid’s elements. According to Heath [Heath,1956,2] (see in particular the comments in relation to propositions 9 and 10), this knowledge can be traced to the school of Pythagoras and therefore comfortably dates before the construction of the theatre at Epidaurus. Ancient Greek mathematicians were always interested in relations between sections satisfying aesthetic criteria and the role of the Golden ratio is central within this context. We therefore claim that the geometrical construction presented in this subsection was known to them. To see incommensurability, Greek mathematics was always interested in relations between sections. In other words, very often Greek mathematicians tried to approximate irrational numbers with rational numbers. Thus, the equations in (17) must have been known to them. These can be expressed in the form of a recurrence relation :

$$\phi^n = \phi \; a_n + b_n \tag{31}$$

where a_n and b_n are integers. Ancient Greek Mathematicians most probably only studied the first few members of the Fibonacci sequence, but the theater in Epidaurus indicates they were aware of the 8^{th} power, i.e. 34 and judging from information embedded in the stones, most probably even reached the 10^{th} power (quite an achievement !). In general, it is easy to prove that a_n and b_n are the Fibonacci numbers, because multiplication of (14) by ϕ^n gives :

$$\phi^{n+2} = \phi^{n+1} + \phi^n \tag{32}$$

which combined with (31) gives the Fibonacci recursion formula for a_n and b_n, namely $a_{n+2} = a_{n+1} + a_n$ and similarly for b_n. This completes the proof based on our "modernized" version of what the Greeks were capable of demonstrating.

Therefore, we can be confident that the ancient Greeks already knew about Fibonacci numbers and this knowledge could well have been transmitted to the Muslims. In hindsight, this is not surprising : both knew the Golden ratio ϕ and from there, it was merely a matter of time, before stumbling onto the Fibonacci sequence. However, we must not forget that our last demonstration concerning Greek thought, was *geometrical*. The Fibonacci numbers as Fibonacci himself presented them, derived from a biological reproductive model - and as we claim - were also discovered by him through such a model. This is closer to the botanical motifs in ancient Egyptian jewelry mentioned before. At any rate, given the ubiquitous nature of the Fibonacci numbers and the Muslim knowledge of algebra, where the "unknown" x of any equation can be disconnected from a geometrical interpretation and become a number representing anything, it is plausible that the intellectual élite of Bejaia was aware of some terms at least in the Fibonacci sequence in the context of bee ancestry and reproduction considering the obvious mercantile impetus of bee apiculture.

4. Conclusions

So far, we have presented a reasonable conjecture as to the origin of the Fibonacci numbers that fits the known historical and mathematical facts. The conjecture states that Fibonacci derived the sequence directly from the intellectual and commercial culture of Bejaia where he was stationed in his formative years prior to 1202 CE. This conjecture further suggests a collaboration with some members of the Toledan "school" of translation, in particular Michael Scotus to whom the revised edition of his *Liber abaci* was dedicated and whose writings and translations contained significant traces of this collaboration. Furthermore, Fibonacci's dedication could also have acted in assuring approval of Fibonacci's book in Christian Europe as Michael Scotus, at that time, was well connected with the papal authorities [Thorndike,1965, p.1]. It was only later that Scotus would be labeled as a wizard. In view of the historical and mathematical data presented here, this conjecture is plausible.

From a historical and chronological point of view, there is an issue concerning how and when Scotus and Fibonacci first met for this collaboration to materialize. This discussion lies outside the score of this investigation. However, this is a minor point as one of the most significant criticisms against this conjecture is the skepticism concerning the possibility that the inhabitants of Bejaia could have worked out the ancestries of bees during the middle-ages. The skepticism is understandable as the subject of genetics has only been developed within the 19th and 20th centuries. To address this issue, we must point out that although the modern science of genetics fully explains the bee ancestries, it is not actually necessary to resort to this in order to understand the ancestry of bees. The observations Aristotle made over 2000 years ago, concerning bee apiculture and bee sexual reproduction coupled with the recognition of the sexes associated with a (matriarchal) 3-member bee caste system are sufficient to realize that a bee drone results from an unfertilized egg. This in turn allows one to tabulate the bee ancestries and obtain the Fibonacci sequence. From a graph theory point of view, this process is sufficient to establish the relationships between the tree nodes of the bee 'family tree" of Figure 3b.

In hindsight, what is conjectured in this article is reasonable. Given the ubiquitous nature of the Fibonacci numbers within nature, it would not have been surprising for someone to observe their presence long before Leonardo de Pisa himself, notwithstanding the issue of written documents discarded or lost over time. The gold bracelet found in the tomb of a king at Abydos (Figure 4) testifies to this fact. Evidence surrounding the theater of Epidaurus in which Fibonacci numbers are embedded strongly suggests that Ancient Greeks were also aware of them. These numbers also derive, as we have demonsrated, from mathematical manipulations accessible to Ancient Greeks. The Golden number and the Fibonacci sequence are mathematical entities inextricably linked and knowledge of one eventually leads to the other. We have mentioned four different locations and periods in which awareness of at least some of the Fibonacci numbers is manifested. In descending order of certainty, we mentioned :

1. Ancient India around 500 BCE in music.
2. Bejaia in Algeria around 1200 CE in bee apiculture.
3. Ancient Greece around the 2nd century BCE in architecture.
4. Archaic Egypt during the 1st dynasty, around 3000 BCE in botanical motifs for jewelry.

We do not claim that knowledge of members of the Fibonacci numbers at any of these four locations necessarily derived from any of the other locations, only that the ubiquitous nature of the Fibonacci sequence allows their discovery in a variety of independent locations at different times and within different contexts. Knowledge of the bee ancestries in Bejaia alone was sufficient for their discovery by Leonardo de Pisa.

We now come full circle to the current of ideas that influence Kepler in the formulation of Kepler's laws : Pythagorean mathematics with its knowledge of the Golden number and the numbers ubiquitous to nature, art and music and embedded in Ancient Greek structures such as the theater of Epidaurus, rediscovered by Fibonacci though his famous rabbit reproduction model. This knowledge was likely transmitted via the corridor of goods, services and ideas through North-Africa to Europe in the middle-ages. We argue in this paper that in the fertile environment generated by this knowledge, a reproductive model reflecting the bee "family trees" may well have directly influenced Leonardo de Pisa thus precipitating the generation of the Fibonacci sequence.

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