

Carl Gustav Jacob Jacobi | Encyclopedia.com

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(*b.* Potsdam, Germany, 10 December 1804; *d.* Berlin, Germany, 18 February 1851)

mathematics.

The second son of Simon Jacobi, a Jewish banker, the precocious boy (originally called Jacques Simon) grew up in a wealthy and cultured family. His brother Moritz, three years older, later gained fame as a physicist in [St. Petersburg](#). His younger brother, Eduard, carried on the banking business after his father's death. He also had a sister, Therese.

After being educated by his mother's brother, Jacobi entered the Gymnasium at Potsdam in November 1816. Promoted to the first (highest) class after a few months in spite of his youth, he had to remain there for four years because he could not enter the university until he was sixteen. When he graduated from the Gymnasium in the spring of 1821, he excelled in Greek, Latin, and history and had acquired a knowledge of mathematics far beyond that provided by the school curriculum. He had studied Euler's *Introductio in analysin infinitorum* and had attempted to solve the general fifth-degree algebraic equation.

During his first two years at the University of Berlin, Jacobi divided his interests among philosophical, classical, and mathematical studies. Seeing that time would not permit him to follow all his interests, he decided to concentrate on mathematics. University lectures in mathematics at that time were at a very elementary level in Germany, and Jacobi therefore in private study mastered the works of Euler, Lagrange, and other leading mathematicians. (Dirichlet, at the same time, had gone to Paris, where Biot, Fourier, Laplace, Legendre, and Poisson were active. Apart from the isolated Gauss at Gottingen, there was no equal center of mathematical activity in Germany.)

In the fall of 1824 Jacobi passed his preliminary examination for *Oberlehrer*, thereby acquiring permission to teach not only mathematics but also Greek and Latin to all high school grades, and ancient and modern history to junior high school students. When—in spite of being of Jewish descent—he was offered a position at the prestigious Joachimsthalsche Gymnasium in Berlin in the following summer, he had already submitted a Ph.D. thesis to the university. The board of examiners included the mathematician E. H. Dirksen and the philosopher Friedrich Hegel. Upon application he was given permission to begin work on the *Habilitation* immediately. Having become a Christian, he was thus able to begin a university career as *Privatdozent* at the University of Berlin at the age of twenty.

Jacobi's first lecture, given during the winter term 1825-1826, was devoted to the analytic theory of curves and surfaces in three-dimensional space. He greatly impressed his audience by the liveliness and clarity of his delivery, and his success became known to the Prussian ministry of education. There being no prospect for a promotion at Berlin in the near future, it was suggested that Jacobi transfer to the University of Königsberg, where a salaried position might be available sooner. When he arrived there in May 1826, the physicists Franz Neumann and Heinrich Dove were just starting their academic careers, and Friedrich Bessel, then in his early forties, occupied the chair of astronomy. Joining these colleagues, Jacobi soon became interested in applied problems. His first publications attracted wide attention among mathematicians. On 28 December 1827 he was appointed associate professor, a promotion in which Legendre's praise of his early work on elliptic functions had had a share. Appointment as full professor followed on 7 July 1832, after a four-hour disputation in Latin. Several months earlier, on 11 September 1831, Jacobi had married Marie Schwinck, the daughter of a formerly wealthy *Kommerzienrat* who had lost his fortune in speculative transactions. They had five sons and three daughters.

For eighteen years Jacobi was at the University of Königsberg, where his tireless activity produced amazing results in both research and academic instruction. Jacobi created a sensation among the mathematical world with his penetrating investigations into the theory of elliptic functions, carried out in competition with Abel. Most of Jacobi's fundamental research articles in the theory of elliptic functions, mathematical analysis, [number theory](#), geometry, and mechanics were published in *Crelle's Journal für die reine und angewandte Mathematik*. With an average of three articles per volume, Jacobi was one of its most active contributors and quickly helped to establish its international fame. Yet his tireless occupation with research did not impair his teaching. On the contrary—never satisfied to lecture along trodden paths, Jacobi presented the substance of his own investigations to his students. He would lecture up to eight or ten hours a week on his favorite subject, the theory of elliptic functions, thus demanding the utmost from his listeners. He also inaugurated what was then a complete novelty in mathematics—research seminars—sembling the more advanced students and attracting his nearest colleagues.

Such were Jacobi's forceful personality and sweeping enthusiasm that none of his gifted students could escape his spell: they were drawn into his sphere of thought, worked along the manifold lines he suggested, and soon represented a "school". C. W. Borchardt, E. Heine, L. O. Hesse, F. J. Richelot, J. Rosenhain, and P. L. von Seidel belonged to this circle; they contributed much to the dissemination not only of Jacobi's mathematical creations but also of the new research-oriented attitude in

university instruction. The triad of Bessel, Jacobi, and Neumann thus became the nucleus of a revival of mathematics at German universities.

In the summer of 1829 Jacobi journeyed to Paris, visiting Gauss in Göttingen on his way and becoming acquainted with Legendre (with whom he had already been in correspondence), Fourier, Poisson, and other eminent French mathematicians. In July 1842 Bessel and Jacobi, accompanied by Marie Jacobi, were sent by the king of Prussia to the annual meeting of the British Association for the Advancement of Science in Manchester, where they represented their country splendidly. They returned via Paris, where Jacobi gave a lecture before the Academy of Science.

Early in 1843 Jacobi became seriously ill with diabetes. Dirichlet, after he had visited Jacobi for a fortnight in April, procured a donation (through the assistance of [Alexander von Humboldt](#)) from Friedrich Wilhelm IV, which enabled Jacobi to spend some months in Italy, as his doctor had advised. Together with Borchardt and Dirichlet and the latter's wife, he traveled in a leisurely manner to Italy, lectured at the science meeting in Lucca (but noticed that none of the Italian mathematicians had really studied his papers), and arrived in Rome on 16 November 1843. In the stimulating company of these friends and of the mathematicians L. Schläöfli and J. Steiner, who also lived in Rome at that time, and further blessed by the favorable climate, Jacobi's health improved considerably. He started to compare manuscripts of Diophantus' *Arithmetica* in the Vatican Library and began to resume publishing mathematical articles. By the end of June 1844 he had returned to Berlin. He was granted royal permission to move there with his family because the severe climate of Königsberg would endanger his health. Jacobi received a bonus on his salary to help offset the higher costs in the capital and to help with his medical expenses. As a member of the Prussian Academy of Sciences, he was entitled, but not obliged, to lecture at the University of Berlin. Because of his poor health, however, he lectured on only a very limited scale.

In the revolutionary year of 1848 Jacobi became involved in a political discussion in the Constitutional Club. During an impromptu speech he made some imprudent remarks which brought him under fire from monarchists and republicans alike. Hardly two years before, in the dedication of volume I of his *Opuscula mathematica* to Friedrich Wilhelm IV, he had expressed his royalist attitude; now he had become an object of suspicion to the government. A petition of Jacobi's to become officially associated with the University of Berlin, and thus to obtain a secure status, was denied by the ministry of education. Moreover, in June 1849 the bonus on his salary was retracted. Jacobi, who had lost his inherited fortune in a bankruptcy years before, had to give up his Berlin home. He moved into an inn and his wife and children took up residence in the small town of Gotha, where life was considerably less expensive.

Toward the end of 1849 Jacobi was offered a professorship in Vienna. Only after he had accepted it did the Prussian government realize the severe blow to its reputation which would result from his departure. Special concessions from the ministry and his desire to stay in his native country finally led Jacobi to reverse his decision. His family, however, was to remain at Gotha for another year, until the eldest son graduated from the Gymnasium. Jacobi, who lectured on [number theory](#) in the summer term of 1850, joined his family during vacations and worked on an astronomical paper with his friend P. A. Hansen.

Early in 1851, after another visit to his family, Jacobi contracted influenza. Hardly recovered, he fell ill with smallpox and died within a week. His close friend Dirichlet delivered the memorial lecture at the Berlin Academy on 1 July 1852, calling Jacobi the greatest mathematician among the members of the Academy since Lagrange and summarizing his eminent mathematical contributions.

The outburst of Jacobi's creativity at the very beginning of his career, combined with his self-conscious attitude, early caused him to seek contacts with some of the foremost mathematicians of his time. A few months after his arrival at Königsberg he informed Gauss about some of his discoveries in number theory, particularly on cubic residues, on which he published a first paper in 1827. Jacobi had been inspired by Gauss's *Disquisitiones arithmeticae* and by a note on the results which Gauss had recently presented to the Göttingen Academy, concerning biquadratic residues. Obviously impressed, Gauss asked Bessel for information on the young mathematician and enclosed a letter for Jacobi, now lost—as are all subsequent letters from Gauss to Jacobi. No regular correspondence developed from this beginning.

Another contact, established by a letter from Jacobi on 5 August 1827, initiated an important regular mathematical correspondence with Legendre that did not cease until Legendre's death. Its topic was the theory of elliptic functions, of which Legendre had been the great master until Abel and Jacobi came on the scene. Their first publications in this subject appeared in September 1827—Abel's fundamental memoir "Recherches sur les fonction elliptiques" in Crelle's *journal* (2, no. 2) and Jacobi's "Extraits de deux lettres ..." in *Astronomische Nachrichten* (6, no. 123). From these articles it is clear that both authors were in possession of essential elements of the new theory. They had developed these independently: Abel's starting point was the multiplication, Jacobi's the transformation, of elliptic functions; both of them were familiar with Legendre's work.

The older theory centered on the investigation of elliptic integrals, that is, integrals of the type where R is a rational function and $f(x)$ is an integral function of the third or fourth degree. Examples of such integrals had been studied by [John Wallis](#), Jakob I and Johann I Bernoulli, and in particular G. C. Fagnano. Euler continued this work by investigating the arc length of a lemniscate, ; by integrating the differential equation

he was led to the addition formula for this integral (elliptic integral of the first kind). When he extended these investigations—for example, to the arc length of an ellipse (elliptic integral of the second kind)—he concluded that the sum of any number of elliptic integrals of the same kind (except for algebraic or logarithmic terms, which may have to be added) may be expressed by a single integral of this same kind, of which the upper limit depends algebraically on the upper limits of the elements of the sum. This discovery shows Euler to be a forerunner of Abel.

The systematic study of elliptic integrals and their classification into the first, second, and third kinds was the work of Legendre, who had cultivated this field since 1786. The leading French mathematicians of his day were interested in the application of mathematics to astronomy and physics. Therefore, although Legendre had always emphasized the applicability of his theories (for instance, by computing tables of elliptic integrals), they did not appreciate his work. Gauss, on the other hand, was well aware of the importance of the subject, for he had previously obtained the fundamental results of Abel and Jacobi but had never published his theory. Neither had he given so much as a hint when Legendre failed to exploit the decisive idea of the inverse function.

It was this idea, occurring independently to both Abel and Jacobi, which enabled them to take a big step forward in the difficult field of transcendental functions. Here Abel's investigations were directed toward the most general question; Jacobi possessed an extraordinary talent for handling the most complicated mathematical apparatus. By producing an almost endless stream of formulas concerning elliptic functions, he obtained his insights and drew his conclusions about the character and properties of these functions. He also recognized the relation of this theory to other fields, such as number theory.

When Legendre first learned of the new discoveries of Abel and Jacobi, he showed no sign of envy. On the contrary, he had nothing but praise for them and expressed enthusiasm for their creations. He even reported on Jacobi's first publications (in the *Astronomische Nachrichten*) to the [French Academy](#) and wrote to Jacobi on 9 February 1828:

It gives me a great satisfaction to see two young mathematicians such as you and him [Abel] cultivate with success a branch of analysis which for so long a time has been the object of my favorite studies and which has not been received in my own country as well as it would deserve. By these works you place yourselves in the ranks of the best analysts of our era.

Exactly a year later Legendre wrote in a letter to Jacobi:

You proceed so rapidly, gentlemen, in all these wonderful speculations that it is nearly impossible to follow you—above all for an old man who has already passed the age at which Euler died, an age in which one has to combat a number of infirmities and in which the spirit is no longer capable of that exertion which can surmount difficulties and adapt itself to new ideas. Nevertheless I congratulate myself that I have lived long enough to witness these magnanimous contests between two young athletes equally strong, who turn their efforts to the profit of the science whose limits they push back further and further.

Jacobi, too, was ready to acknowledge fully the merits of Abel. When Legendre had published the third supplement to his *Traité des fonctions elliptiques et des intégrales eulériennes*, in which he presented the latest developments, it was Jacobi who reviewed it for Crelle's *Journal* (8 [1832], 413-417):

Legendre to the transcendental functions, where X exceeds the fourth degree, gives the name "hyperelliptical" [*ultra-elliptiques*]. We wish to call them *Abelsche Transcendenten* (Abelian transcendental functions), for it was Abel who first introduced them into analysis and who first made evident their great importance by his far-reaching theorem. For this theorem, as the most fitting monument to this extraordinary genius, the name "Abelian theorem" would be very appropriate. For we happily agree with the author that it carries the full imprint of the depth of his ideas. Since it enunciates in a simple manner, without the vast setup of mathematical formalism, the deepest and most comprehensive mathematical thought, we consider it to be the greatest mathematical discovery of our time although only future—perhaps distant—hard work will be able to reveal its whole importance.

Jacobi summarized his first two years' research, a good deal of which had been obtained in competition with Abel, in his masterpiece *Fundamenta nova theoriae functionum ellipticarum*, which appeared in April 1829. His previous publications in *Astronomische Nachrichten* and in Crelle's *Journal* were here systematically collected, greatly augmented, and supplemented by proofs—he had previously omitted these, thereby arousing the criticism of Legendre, Gauss, and others.

The *Fundamenta nova* deals in the first part with the transformation, and in the second with the representation, of elliptic functions. Jacobi took as his starting point the general elliptic differential of the first kind and reduced it by a second-degree transformation to the normal form of Legendre. He studied the properties of the functions U (even) and V (odd) in the rational transformations $Y=U/V$ and gave as examples the transformations of the third and fifth degrees and the pertinent modular equations. By combining two transformations he obtained the multiplication of the elliptic integral of the first kind, a remarkable result. He then introduced the inverse function $\phi = \text{am } u$ into the elliptic integral

hence

$$x = \sin \phi = \sin \text{am } u.$$

Further introducing $\cos am u = \text{am}(K - u)$

he collected a large number of formulas. Using the substitution $\sin \phi = i \tan \psi$, he established the relation

$$\sin am(iu, k) = i \tan am(u, k')$$

the moduli k and k' are connected by the equation $k^2 + k'^2 = 1$. He thus obtained the double periodicity, and the zero values, the infinity values, and the change of value in half a period for the elliptic functions. This introduction of the imaginary into the theory of elliptic functions was another very important step results is the demonstration of the invariance of the modular equations when the same transformation is applied to the primary and secondary moduli. Toward the end of the first part of his work Jacobi developed the third-order differential equations which is satisfied by all transformed moduli.

The second part of the *Fundamenta nova* is devoted to the evolution of elliptic function into infinite products and series of various kinds. The first representation of the elliptic functions $\sin am u$, $\cos am u$, $\Delta am u$, which he gave is in the form of quotients of infinite products. Introducing q , Jacobi expressed the modulus and periods in terms of q , as for instance

Another representation of the elliptic functions and their n th powers as Fourier series leads to the sums (in terms of the moduli) of various infinite series in q . Integrals of the second kind are treated after the function.

has been introduced. Jacobi reduced integrals of the third kind to integrals of the first and second kinds and a third transcendental function which also depends on two variables only. In what follows, Jacobi's function.

played a central role. It is then supplemented by the function $H(u)$ such that $\sin am$.

$\Theta(u)$ and $H(u)$ are represented as infinite products and as Fourier series. The latter yield such remarkable formulas as

After a number of further summations and identities Jacobi closed this work with an application to the theory of numbers. From the identity.

$$= 1 + 8 \sum \phi(p)(q^p + 3q^{2p} + 3q^{4p} + 3q^{8p} + \dots),$$

where $\phi(p)$ is the sum of the divisors of the odd number p , he drew the conclusion that any integer can be represented as the sum of at most four squares, as Fermat had suggested.

Jacobi lectured on the theory of elliptic functions for the first time during the winter term 1829-1830, emphasizing that double periodicity is the essential property of these functions. The theta function should be taken as foundation of the theory; the representation in series with the general term $e^{-(an+b)^2}$ ensures convergence and makes it possible to develop the whole theory. In his ten hours a week of lecturing in the winter of 1835-1836 Jacobi for the first time founded the theory on the theta function, proving the famous theorem about the sum of products of four theta functions and defining the kinds of elliptic functions as quotients of theta functions. He continued this work in his lectures of 1839-1840, the second part of which is published in volume I of his *Gesammelte Werke*. volume II contains a historical summary, "Zur Geschichte der elliptischen und Abel'schen Transcendenten," composed by Jacobi probably in 1847, which documents his view of his favorite subject toward the end of his life.

Some of Jacobi's discoveries in number theory have already been mentioned. Although he intended to publish his results in book form, he was never able to do so. The theory of residues, the division of the circle into n equal parts, the theory of quadratic forms, the representation of integers as sums of squares or cubes, and related problems were studied by Jacobi. During the winter of 1836-1837 he lectured on number theory, and some of his methods became known through Rosenhain's lecture notes. In 1839 Jacobi's *Canon arithmeticus* on primitive roots was published; for each prime and power of a prime less than 1,000 it gives two companion tables showing the numbers with given indexes and the index of each given number.

Most of Jacobi's work is characterized by linkage of different mathematical disciplines. He introduced elliptic functions not only into number theory but also into the theory of integration, which in turn is connected with the theory of differential equations where, among other things, the principle of the last multiplier is due to Jacobi. Most of his investigations on first-order partial differential equations and analytical mechanics were published posthumously (in 1866, by Clebsch) as *Vorlesungen über Dynamik*. Taking W. R. Hamilton's research on the differential equations of motion (canonical equations) as a starting point, Jacobi also carried on the work of the French school (Lagrange, Poisson, and others). He sought the most general substitutions that would transform canonical differential equations into such equations. The transformations are to be such that a canonical differential equation (of motion) is transformed into another differential equation which is again canonical. He also developed a new theory for the integration of these equations, utilizing their relation to a special Hamiltonian differential equation. This methods enabled him to solve several very important problems in mechanics and astronomy. In some special cases Clebsch later improved Jacobi's results, and decades later Helmholtz carried Jacobi's mechanical principles over into physics in general.

Among Jacobi's work in mathematical physics is research on the attraction of ellipsoids and a surprising discovery in the theory of configurations of rotating liquid masses. Maclaurin had shown that a homogeneous liquid mass may be rotated uniformly about a fixed axis without change of shape if this shape is an ellipsoid of revolution. D'Alembert, Laplace, and Lagrange had studied the same problem; but it was left for Jacobi to discover that even an ellipsoid of three different axes may satisfy the conditions of equilibrium.

The theory of determinants, which begins with Leibniz, was presented systematically by Jacobi early in 1841. He introduced the "Jacobian" or functional determinant; a second paper—also published in Crelle's *Journal*—is devoted entirely to its theory, including relations to inverse functions and the transformation of multiple integrals.

Jacobi was also interested in the history of mathematics. In January 1846 he gave a public lecture on Descartes which attracted much attention. In the same year A. von Humboldt asked him for notes on the mathematics of the ancient Greeks as material for his *Kosmos* and Jacobi readily complied—but Humboldt later confessed that some of the material went beyond his limited mathematical knowledge. In the 1840's Jacobi became involved in the planning of an edition of Euler's works. He corresponded with P. H. von Fuss, secretary of the [St. Petersburg](#) Academy and great-grandson of the famous mathematician, who had discovered a number of Euler's unpublished papers. Jacobi drew up a very detailed plan of distributing the immense number of publications among the volumes of the projected edition. Unfortunately, the project could be realized only on a much reduced scale. It was not until 1911 that the first volume of *Leonhardi Euleri opera omnia*—still in progress—appeared.

Jacobi's efforts to promote an edition of Euler were prompted by more than the ordinary interest a mathematician might be expected to take in the work of a great predecessor. Jacobi and Euler were kindred spirits in the way they created their mathematics. Both were prolific writers and even more prolific calculators; both drew a good deal of insight from immense algorithmical work; both labored in many fields of mathematics (Euler, in this respect, greatly surpassed Jacobi); and both at any moment could draw from the vast armory of mathematical methods just those weapons which would promise the best results in the attack on a given problem. Yet while Euler divided his energies about equally between pure and applied mathematics, Jacobi was more inclined to investigate mathematical problems for their intrinsic interest. Mathematics, as he understood it, had a strong Platonic ring. For the disputation at his inauguration to a full professorship in 1832 Jacobi had chosen as his first thesis "Mathesis est scientia eorum, quae per se clara sunt."

BIBLIOGRAPHY

1. Original Works. Jacobi's works have been collected twice. *Opuscula mathematica* is in 3 vols. (Berlin, 1846-) Vol. I was edited by Jacobi himself; vol. II, also prepared by him, was published posthumously by Dirichlet; vol. III was published by his pupil C. W. Borchardt.

The standard ed., 7 vols. and supp., was issued by the Prussian Academy of Sciences as *C. G. J. Jacobi's Gesammelte Werke*, C. W. Borchardt, A. Clebsch, and K. Weierstrass, eds. (Berlin, 1881-1891). Vol I contains, among other works, the *Fundamenta nova theoriae functionum ellipticarum* (Königsberg, 1829). The supp. vol. is *Vorlesungen über Dynamik*, first published by A. Clebsch (Leipzig, 1866) *Gesammelte Werke* has been repr. ([New York](#), 1969).

Jacobi's only other publication in book form, the *Canon arithmeticus* (Berlin, 1839), is not in the *Gesammelte Werke* but appeared in a 2nd ed. recomputed by W. Patz and edited by H. Brandt (Berlin, 1956).

Kurt-R. Biermann has published "Eine unveröffentlichte Jugendarbeit C. G. J. Jacobis über wiederholte Funktionen," in *Journal für die reine und angewandte Mathematik*, **207** (1961), 96-112.

A list of Jacobi's publications and of his lectures is in *Gesammelte Werke*, VII, 4214-440. See also Poggendorff, I, 1178-1181, 1576; III, 681; IV, 688; VIIa, Supp. 302-303.

Brief information on 16 vols. of manuscript material, in the archives of the Deutsche Akademie der Wissenschaften in Berlin, is in *Gelehrten- und Schriftstellernachlässe in den Bibliotheken der DDR*, I (Berlin, 1959), 50, no.315, "Jakobi"[sic].

II. Secondary Literature. The main secondary sources are J. P. G. Lejeune Dirichlet, "Gedächtnisrede" (1852), repr. in *Gesammelte Werke*, I; and Leo Koeningsberger, *Carl Gustav Jacob Jacobi. Festschrift zur Feier der hundertsten Wiederkehr seines Geburtstages* (Leipzig, 1904); and *Carl Gustav Jacob Jacobi. Rede zu der von dem Internationalen Mathematiker-Kongress in Heidelberg veranstalteten Feier der hundertsten Wiederkehr seines Geburtstages, gehalten am 9. August 1904* (Leipzig, 1904), also in *Jahresbericht der Deutschen Mathematikervereinigung*, 13 (1904), 405-433. For further Secondary Literature see Poggendorff, esp. VIIa Supp.

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