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(*b*. Beauvais France, 28 June 1875; *d* Paris, France, 26 July 1941)

mathematics.

Lebesgue studied at the École Normale Supėrieure from 1894 to 1897. His first university positions were at Rennes (1902—1906) and Poitiers (1906—1910). At the Sorbonne, he became *maitre de conférences* (in mathematical analysis, 1910—1919) and then *professeur d'application de la géométrie à I'analyse*. In 1921 he was named professor at the Collège de France and the following year was elected to the Académie des Sciences.

Lebesgue's outstanding contribution to mathematics was the theory of integration that bears his name and that became the foundation for subsequent work in integration theory and its applications. After completing his studies at the École Normale Supérieure, Lebesgue spent the next two years working in its library, where he became acquainted with the work of another recent graduate, René Baire. Baire's unprecedented and successful researches on the theory of discontinuous functions of a real variable indicated to Lebesgue what could be achieved in this area. From 1899 to 1902, while teaching at the Lycée Centrale in Nancy, Lebesgue developed the ideas that he presented in 1902 as his doctoral thesis at the Sorbonne. In this work Lebesgue began to develop his theory of integration which, as he showed, includes within its scope all the bounded discontinuous functions introduced by Baire.

The Lebesgue integral is a generalization of the integral introduced by Riemann in 1854. As Riemann's theory of integration was developed during the 1870's and 1880's a measure-theoretic viewpoint was gradually introduced. This viewpoint was made especially prominent in Camille Jordan's treatment of the Riemann integral in his *Cours d'analyse* (1893) and strongly influenced Lebesgue's outlook on these matters. The significance of placing integration theory within a measure-theoretic context was that it made it possible to see that a generalization of the notions of measure and measurability carries with it correspondilng generalizations of the notions of the integral and integrability. In 1989 Émile Borel was led through radically different notions of measure and measurability. Some mathematicians found Borel's ideas lacking in appeal and relevance, especially since they involved assigning measure zero to some dense sets. Lebesgue, however, accepted them. He completed Borel's definitions of measure and measurability so that they represented generalizations of Jordan's definitions and then used them to obtain his generalization of the Riemann integral.

After the work of Jordan and Borel, Lebesgue's generalizations were somewhat inevitable. Thus, W. H. Young and G. Vitali, independently of Lebesgue and of each other, introduced the same generalization of Jordan's theory of measure; in Young's case, it led to a generalization of the integral that was essentially the same as Lebesgue's. In Lebesgue's work, however, the generalized definition of the integral was simply the starting point of his contributions to integration theory. What made the new definition important was that Lebesgue was able to recognize in it and analytical tool capable of dealing with-and to a large extent overcoming—the numerous theoretical difficulties that had arisen in connection with Riemann's theory of integration. In fact, the problems posed by these difficulties motivated all of Lebesgue's major results.

The first such problem had been raised unwittingly by Fourier early in the nineteenth century: (1) If a bounded foundation can be represented by a trigonometric series, is that series the Fourier series of the function? Closely related to (1) is (2): When is the term-by-term integration of and infinite series permissible? Fourier had assumed that for bounded functions the answer to (2) is Alwasys, and he had used this assumption to prove that the answer to (1)is Yes.

By the end of the nineteenth century it was recognized—and emphasized—that term-by-term integration is not always valid even for uniformly bounded series of Riemann-integrable functions, precisely because the function represented by the series need not be Riemann-integrable. These developments, however, paved the way for Lebesgue's elegant proof that term-by-term integration is permissible for any uniformly bounded series of Lebesgue-integrable functions. By applying this result to (1), Lebesgue was able to affirm Fourier's conclusion that the answer is Yes.

Another source of difficulties was the fundamental theorem of the calculus,

The work of Dini and volterra in the period 1878-1881 made it clear that functions exist which have bounded derivatives that are not integrable in Riemann's sense, so that the fundamental theorem becomes meaningless for these functions. Later further classes of these functions were discovered; and additional problems arose in connection with Harnack's extension of the Riemann integral to unbounded functions because continuous functions with densely distributed intervals of invariability were discovered. These functions provided examples of Harnack-integrable derivatives for which the fundamental theorem is false. Lebesgue showed that for bounded derivatives these difficulties disappear entirely when integrals are taken in his sense. He also showed that the fundamental theorem is true for and unbounded, finite-valued derivative f'that is Lebesgue-integrable and that this is the case if, and only if, f is of bounded variation. Furtermore, Lebesgue's suggestive observations concerning the case in which f'is finite-valued but not Lebesgue- integrable were successfully developed by Arnaud Denjoy, starting in 1912, using the transfinite methods developed by Baire.

The discovery of continuous monotonic functions with densely distributed intervals of invariability also raised the question: When is a continuous function and integral? The question prompted Harnack to introduce the property that has since been termed absolute continuity. During the 1890's absolute continuity came to be regarded as the characteristic property of absolutely convergent integrals, although no one was actually able to show that every absolutely continuous function is and integral. Lebesgue, however, perceived that this is precisely the case when integrals are taken in his sense.

A deeper familiarity with infinite sets of points had led to the discovery of the problems connected with the fundamental theorem. The nascent theory of infinite sets also stimulated and interest in the meaning fulness of the customary formula

for the length of the curve y = f(x). Paul du Bois Reymond, who initiated and interest in the problem in 1879, was convinced that the theory of integration is indispensable for the treatment of the concepts of rectifiability and curve length within the general context of the modern function concept. But by the end of the nineteenth century this view appeared untenable, particularly because of the criticism and counterexamples given by Ludwig Scheeffer. Lebesgue was quite interested in this matter and was able to use the methods and results of his theory of integration to reinstate the credibility of du Bois-Reymond's assertion that the concepts of curve length and integral are closely related.

Lebesgue's work on the fundamental theorem and on the theory of curve rectification played and important role in his discovery that a continuous function of bounded variation possesses a finite derivative except possibly on a set of Lebesgue measure zero. This theorem gains in significance when viewed against the background of the century-long discussion of the differentiability properties of continuous functions. During roughly the first half of the nineteenth century, it was generally though that continuous functions are differentiable at "most"points, although continuous functions were frequently assumed to be "piecewise"monotonic. (Thus, differentiability and monotonicity were linked together, albeit tenuously.) By the end of the century this view was discredited, and no less a mathematician than Weierstrass felt that there must exist continuous monotonic functions that are nowhere differentiable. Thus, in a sense, Lebesgue's theorem substantiated the intuitions of and earlier generation of mathematicians.

Riemann's definition of the integral also raised problems in connection with the traditional theorem positing the identity of double and iterated integrals of a function of two variables. Examples were discovered for which the theorem fails to hold. As a result, the traditional formulation of the theorem had to be modified, and the modifications became drastic when Riemann's definition was extended to unbounded functions. Although Lebesgue himself did not resolve this infelicity, it was his treatment of the problem that formed the foundation for Fubini's proof (1907) that the Lebesgue integral does make it possible to restore to the theorem the simplicity of its traditional formulation.

During the academic years 1902-1903 and 1904-1905, Lebesgue was given the honor of presenting the Cours Peccot at the Collège de France. His lectures, published as the monographs *Leçons sur l'intégration … (1904) and Leç sur les séries trigonométriques* (1906), served to make his ideas better known. By 1910 the number of mathematicians engaged in work involving the Lebesgue integral began to increase rapidly. Lebesgue's own work—particularly his highly successful applications of his integral in the theory of trigonometric series—was the chief reason for this increase, but the pioneering research of others, notably Fatou, F. Riesz, and Fischer, also contributed substantially to the trend. In particular, Riesz's work on L^p spaces secured a permanent place for the Lebesgue integral in the theory of integral equations and function spaces.

Although Lebesgue was primarily concerned with his own theory of integration, he played a role in bringing about the development of the abstract theories of measure and integration that predominate in contemporary mathematical research. In 1910 he published and important memoir, "Surl'intégration des fonctions discontinues,"in which he extended the theory of integration and differentiation to *n*-dimensional space. Here Lebesgue introduced, and made fundamental, the notion of a countably additive set function (defined on Lebesgue-measurable sets) and observed in passing that such functions are of bounded variation on sets on which they take a finite value. By thus linking the notions of bounded variation and additivity, Lebesgue's observation suggested to Radon a definition of the integral that would include both the definitions of Lebesgue and Stieltjes as special cases. Radon's paper (1913), which soon led to further abstractions, indicated the viability of Lebesgue's ideas in a much more general setting.

By the time of his election to the Académie des Sciences in 1922, Lebesgue had produced nearly ninety books and papers, Much of this output was concerned with his theory of integration, but he also did significant work on the structure of sets and functions (later carried further by Lusin and others), the calculus of variations, the theory of surface area, and dimension theory.

For his contributions to mathematics Lebesgue received the Prix Houllevigue (1912), the Prix Poncelet (1914). the Prix Saintour (1917), and the Prix petit d'Ormoy (1919), During the last twenty years of his life, he remained very active, but his writings reflected a broadening of interests and were largely concerned with pedagogical and historical questions and with elementary geometry.

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